SCAN 2 — Solution of Math Test #4

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Exercise 1. We know that

$$e^x \underset{x\to 0}{=} 1 + x + \frac{x^2}{2} + o\big(x^2\big)$$

hence, since $(-1)^n/n^\alpha \underset{n\to+\infty}{\longrightarrow} 0$,

$$\exp\left(\frac{(-1)^n}{n^\alpha}\right) - 1 \underset{n\to+\infty}{=} \frac{(-1)^n}{n^\alpha} + \frac{1}{n^{2\alpha}} + o\left(\frac{1}{n^2}\right).$$

We can write the general term of the series as

$$\forall n \in \mathbb{N}^*,\ \exp\left(\frac{(-1)^n}{n^\alpha}\right) - 1 = v_n + w_n$$

with

$$\forall n \in \mathbb{N}^*,\ v_n = \frac{(-1)^n}{n^\alpha},\ w_n = \exp\left(\frac{(-1)^n}{n^\alpha}\right) - 1 - v_n.$$

Now, we know that the series $\sum_n v_n$ converges (it's an alternating Riemann series), and since

$$w_n \underset{n\to+\infty}{=} \frac{1}{n^{2\alpha}} + o\left(\frac{1}{n^{2\alpha}}\right) \underset{n\to+\infty}{\sim} \frac{1}{n^{2\alpha}} > 0$$

we conclude that the series $\sum_n w_n$ converges if and only if $2\alpha > 1$, i.e., if and only if $\alpha > 1/2$.
We know that the sum of a convergent series and a divergent series yields a divergent series, and that the sum of two convergent series yields a convergent one.
Hence, the series converges if and only if $\alpha > 1/2$.

Exercise 2.

1. $$\frac{n}{1+n^3} \underset{n\to+\infty}{\sim} \frac{1}{n^2} > 0$$

   and by Riemann (with $\alpha = 2 > 1$), the series $\sum_n 1/n^2$ converges. Hence, by the equivalent test, the series $\sum_n n/\big(1+n^3\big)$ converges.

2. Let $t \in (0,+\infty)$. Since $1 + t^3 > t^3 > 0$,

   $$\frac{1}{1+t^3} < \frac{1}{t^3}$$

   and multiplying by $t > 0$ yields the first inequality.

   Let $N \in \mathbb{N}^*$ and let $t \in [N+1,+\infty)$. Then

   $$\frac{t}{1+t^3} - \frac{1}{\left(1+\dfrac{1}{(N+1)^3}\right)t^2} = \frac{\left(1+\dfrac{1}{(N+1)^3}\right)t^3 - 1 - t^3}{t^2\left(1+\dfrac{1}{(N+1)^2}\right)t^2} = \frac{\dfrac{t^3}{(N+1)^3} - 1}{t^2\left(1+\dfrac{1}{(N+1)^2}\right)t^2} \geq 0.$$

3. Let $N \geq 2$. Define the function

   $$f : \begin{array}{ccc} [N,+\infty) & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \dfrac{t}{1+t^3}. \end{array}$$

   For $t \in [N,+\infty)$,

   $$f'(t) = \frac{1-2t^3}{\big(1+t^3\big)^2} < 0$$

since $t \geq N \geq 1$, hence the function f is decreasing: we can use the integral comparison test, and we obtain:

$$\frac{1}{1+\dfrac{1}{(N+1)^3}}\int_{N+1}^{+\infty}\frac{\mathrm{d}t}{t^2} \leq R_N = \sum_{n=N+1}^{+\infty}\frac{n}{1+n^3} \leq \int_N^{+\infty}\frac{\mathrm{d}t}{t^2},$$

hence

$$\frac{(N+1)^2}{(N+1)^3+1} \leq R_N \leq \frac{1}{N^2}.$$

Now, the result follows from the fact that $R_N = S - S_N$.

4. From the given numerical values, we obtain (using the previous inequality with $N = 38$)

$$\underline{1.111}3 < S_{38} + \frac{39^2}{39^3+1} \leq S \leq S_{38} + \frac{1}{38} < \underline{1.111}99,$$

hence we obtain the numerical value of S correct to 3 decimal places:

$$S = 1.111\ldots$$

Exercise 3.

1. Let $z \in \mathbb{C}^*$. We use the ratio test:

$$\left|\frac{\dfrac{(n+1)^{n+1}z^{n+1}}{(n+1)!}}{\dfrac{n^n z^n}{n!}}\right| = \frac{(n+1)^n}{n^n}|z| = \left(1+\frac{1}{n}\right)^n |z| \underset{n\to+\infty}{\longrightarrow} \mathrm{e}|z|.$$

Hence, by the ratio test, the power series converges for $|z| < \mathrm{e}^{-1}$ and diverges for $|z| > \mathrm{e}^{-1}$. We conclude that the radius of convergence of the power series is e^{-1}.

2. a) We know that the radius of convergence of $\sum_n na_n x^n$ is R. Now, for $z \in \mathbb{C}$ such that $|z| > \sqrt{R}$, the series $\sum_n a_n z^{2n} = \sum_n a_n\left(z^2\right)^n$ diverges, and for $z \in \mathbb{C}$ such that $|z| < \sqrt{R}$, the series $\sum_n a_n z^{2n} = \sum_n a_n\left(z^2\right)^n$ converges. We hence conclude that $R_g = \sqrt{R}$.

   b) We know that for all $x \in (-R, R)$,

$$f'(x) = \sum_{n=1}^{+\infty} na_n x^{n-1}.$$

   Hence, for $x \in (-R_g, R_g)$,

$$f'\left(x^2\right) = \sum_{n=1}^{+\infty} na_n x^{2n-2},$$

   and we conclude that

$$g(x) = \sum_{n=0}^{+\infty} na_n x^{2n+1} = \sum_{n=1}^{+\infty} na_n x^{2n+1} = x^3\sum_{n=1}^{+\infty} na_n x^{2n-2} = x^3 f'\left(x^2\right).$$

Exercise 4.

1. a) Let $x \in (-R, R)$. We know that f can be differentiated term by term as much as we want within $(-R, R)$, hence

$$f'(x) = \sum_{n=1}^{+\infty} na_n x^{n-1} = \sum_{n=0}^{+\infty}(n+1)a_{n+1}x^n,$$

$$f''(x) = \sum_{n=2}^{+\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{+\infty}(n+2)(n+1)a_{n+2}x^n.$$

   Hence,

$$x^3 f''(x) = \sum_{n=0}^{+\infty} n(n-1)a_n x^{n+1} = \sum_{n=1}^{+\infty}(n-1)(n-2)a_{n-1}x^n,$$

$$x f''(x) = \sum_{n=0}^{+\infty} (n+2)(n+1)a_{n+2}x^{n+1} = \sum_{n=1}^{+\infty} (n+1)na_{n+1}x^n,$$
$$x^2 f'(x) = \sum_{n=0}^{+\infty} na_n x^{n+1} = \sum_{n=1}^{+\infty} (n-1)a_{n-1}x^n,$$

hence

$$\begin{aligned} &x\big(x^2+1\big)f''(x) + \big(x^2-1\big)f'(x) \\ &\qquad = -a_1 + \sum_{n=1}^{+\infty} \Big((n-1)(n-2)a_{n-1} + (n+1)na_{n+1} + (n-1)a_{n-1} - (n+1)a_{n+1} \Big) x^n \\ &\qquad = -a_1 + \sum_{n=1}^{+\infty} \Big((n-1)^2 a_{n-1} + (n+1)(n-1)a_{n+1} \Big) x^n \end{aligned}$$

b) By the identity theorem, f is a solution of Equation (E) on $(-R, R)$ if and only if

$$\begin{cases} a_1 = -1 \\ \forall n \geq 1,\ (n-1)^2 a_{n-1} + (n+1)(n-1)a_{n+1} = 0. \end{cases}$$

Note that the case $n = 1$ is always fulfilled, and if $n \neq 1$ we can simplify by $n-1$. We hence conclude that f is a solution of Equation (E) if and only if:

$$\begin{cases} a_1 = -1 \\ \forall n \geq 2,\ (n-1)a_{n-1} + (n+1)a_{n+1} = 0. \end{cases}$$

2. a) The coefficients (a_n) hence satisfy

$$\forall n \geq 2,\ a_{n+1} = -\frac{n-1}{n+1} a_{n-1}.$$

- We thus have, for the odd ones:

$$a_3 = \frac{1}{3},\ a_5 = -\frac{1}{5},\ a_7 = \frac{1}{7},\ \ldots$$

  and we conjecture that

$$\forall k \in \mathbb{N},\ a_{2k+1} = (-1)^{k+1} \frac{1}{2k+1}.$$

  We check this conjecture by induction: for $k = 0$ the result holds true; assume it true for some $k \in \mathbb{N}$, then

$$a_{2k+3} = -\frac{2k+1}{2k+3} a_{2k+1} = -\frac{2k+1}{2k+3} (-1)^{k+1} \frac{1}{2k+1} = (-1)^{k+2} \frac{1}{2k+3}.$$

- Similarly for the even ones:

$$a_4 = -\frac{1}{2} a_2,\ a_6 = \frac{1}{3} a_2,\ a_8 = -\frac{1}{4} a_2,\ \ldots$$

  and we conjecture that

$$\forall k \in \mathbb{N}^*,\ a_{2k} = (-1)^{k+1} \frac{1}{k} a_2.$$

  We check this conjecture by induction: for $k = 1$ the result holds true; assume it true for some $k \in \mathbb{N}^*$, then

$$a_{2k+2} = -\frac{2k}{2k+2} a_{2k} = -\frac{2k}{2k+2} (-1)^{k+1} \frac{1}{k} a_2 = (-1)^{k+2} \frac{1}{k+1} a_2.$$

b) We split the power series defining f into its odd and even components:

$$f_{\text{even}}(x) = \sum_{k=0}^{+\infty} a_{2k} x^{2k} = a_0 + a_2 \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k} x^{2k},$$
$$f_{\text{odd}}(x) = \sum_{k=0}^{+\infty} a_{2k+1} x^{2k+1} = \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{2k+1} x^{2k+1}.$$

It's now easy to see, e.g., using the ratio test, that the radius of convergence of f_{even} and of f_{odd} is 1 (unless $a_2 = 0$ in which case the radius of convergence of f_{even} is 0). Hence $R = 1$.

3. a)
$$\forall x \in (-1,1),\ h(x) = \frac{1}{1+x} = \sum_{n=0}^{+\infty} (-1)^n x^n.$$
   b) Since a power series can be integrated term by term within its open interval of convergence, we conclude:
$$\forall x \in (-1,1),\ F(x) = \ln(1+x) = \int_0^x \frac{\mathrm{d}t}{1+t} = \int_0^x \left(\sum_{n=0}^{+\infty} (-1)^n t^n\right) \mathrm{d}t = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^n}{n}.$$
   Moreover,
$$\forall x \in (-1,1),\ h(x^2) = \frac{1}{1+x^2} = \sum_{n=0}^{+\infty} (-1)^n x^{2n},$$
   and hence,
$$\forall x \in (-1,1),\ G(x) = \arctan(x) = \int_0^x \frac{\mathrm{d}t}{1+t^2} = \int_0^x \left(\sum_{n=0}^{+\infty} (-1)^n t^{2n}\right) \mathrm{d}t = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$
   The radius of convergence of F and G is that of h, i.e., 1.

4. We hence conclude that:
$$\forall x \in (-1,1),\ f_{\text{even}}(x) = a_0 + a_2 \ln(1+x^2), \qquad \text{and} \qquad f_{\text{odd}}(x) = -\arctan(x).$$
   Hence, the solutions of Equation (E) that possess a power series expansion are of the form
$$f(x) = a_0 + a_2 \ln(1+x^2) - \arctan(x).$$

Exercise 5.

1. Let $z \in \mathbb{C}^*$. We use the ratio test:
$$\left| \frac{\frac{(n+1)^{n+1} z^{n+1}}{(n+1)!}}{\frac{n^n z^n}{n!}} \right| = \frac{(n+1)^n}{n^n} |z| = \left(1 + \frac{1}{n}\right)^n |z| \underset{n\to+\infty}{\longrightarrow} \mathrm{e}|z|.$$
   Hence the radius of the power series is e^{-1}.

2. a) We know that the radius of convergence of $\sum_n n a_n x^n$ is R. Now, for $z \in \mathbb{C}$ such that $|z| > \sqrt{R}$, the series $\sum_n a_n z^{2n} = \sum_n a_n (z^2)^n$ diverges, and for $z \in \mathbb{C}$ such that $|z| < \sqrt{R}$, the series $\sum_n a_n z^{2n} = \sum_n a_n (z^2)^n$ converges. We hence conclude that $R_g = \sqrt{R}$.
   b) We know that for all $x \in (-R, R)$,
$$f'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1}.$$
   Hence, for $x \in (-R_g, R_g)$,
$$f'(x^2) = \sum_{n=1}^{+\infty} n a_n x^{2n-2},$$
   and we conclude that
$$x^3 f'(x^2) = \sum_{n=1}^{+\infty} n a_n x^{2n+1} = \sum_{n=0}^{+\infty} n a_n x^{2n+1} = g(x).$$

Exercise 6. We know that
$$\mathrm{e}^x \underset{x\to0}{=} 1 + x + \frac{x^2}{2} + o(x^2)$$
hence, since $(-1)^n/n^\alpha \underset{n\to+\infty}{\longrightarrow} 0$ since $\alpha > 0$,
$$\exp\left(\frac{(-1)^n}{n^\alpha}\right) - 1 \underset{n\to+\infty}{=} \frac{(-1)^n}{n^\alpha} + \frac{1}{n^{2\alpha}} + o\left(\frac{1}{n^2}\right).$$

We can write the general term of the series as

$$\forall n \in \mathbb{N}^*,\ \exp\left(\frac{(-1)^n}{n^\alpha}\right) - 1 = v_n + w_n$$

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Now, we know that the series $\sum_n v_n$ converges (it's an alternating Riemann series with $\alpha > 0$), and since

$$w_n \underset{n\to+\infty}{=} \frac{1}{n^{2\alpha}} + o\left(\frac{1}{n^{2\alpha}}\right) \underset{n\to+\infty}{\sim} \frac{1}{n^{2\alpha}} > 0$$

we conclude, by the equivalent test, that the series $\sum_n w_n$ converges if and only if $2\alpha > 1$, i.e., if and only if $\alpha > 1/2$.
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Exercise 7.

1. $$\frac{n}{1+n^3} \underset{n\to+\infty}{\sim} \frac{1}{n^2} > 0$$

   and by Riemann, the series $\sum_n 1/n^2$ converges. Hence, by the equivalent test, the series $\sum_n n/\left(1+n^3\right)$ converges.

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   and multiplying by $t > 0$ yields the first inequality.

   Let $N \in \mathbb{N}^*$ and let $t \in [N+1, +\infty)$. Then

   $$\frac{t}{1+t^3} - \frac{1}{\left(1+\dfrac{1}{(N+1)^3}\right)t^2} = \frac{\left(1+\dfrac{1}{(N+1)^3}\right)t^3 - 1 - t^3}{t^2\left(1+\dfrac{1}{(N+1)^2}\right)t^2} = \frac{\dfrac{t^3}{(N+1)^3} - 1}{t^2\left(1+\dfrac{1}{(N+1)^2}\right)t^2} \geq 0.$$

3. Let $N \geq 2$. Define the function

   $$\begin{array}{rccl} f\ : & [N-1, +\infty) & \longrightarrow & \mathbb{R} \\ & t & \longmapsto & \dfrac{t}{1+t^3}. \end{array}$$

   For $t \in [N-1, +\infty)$,

   $$f'(t) = \frac{1-2t^3}{\left(1+t^3\right)^2} < 0$$

   since $t \geq N-1 \geq 1$, hence the function f is decreasing: we can use the integral comparison test, and we obtain:

   $$\frac{1}{1+\dfrac{1}{(N+1)^3}}\int_{N+1}^{+\infty}\frac{\mathrm{d}t}{t^2} \leq R_N = \sum_{n=N+1}^{+\infty}\frac{n}{1+n^3} \leq \int_N^{+\infty}\frac{\mathrm{d}t}{t^2},$$

   hence

   $$\frac{(N+1)^2}{(N+1)^3+1} \leq R_N \leq \frac{1}{N^2}.$$

   Now, the result follows from the fact that $R_N = S - S_N$.

4. From the given numerical values, we obtain (using the previous inequality with $N = 38$)

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Exercise 8.

1.
$$\forall (x,y,z) \in \mathbb{R}^3,\ q(x,y,z) = x^2 + 4xy + 6xz + 2y^2 + 8yz - z^2.$$

2.
$$M = \begin{pmatrix} 1 & 1 & 3/2 \\ 1 & -1 & 0 \\ 3/2 & 0 & -3 \end{pmatrix}.$$