INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — S4— Math Test #4 — 2h

March 30, 2017

No documents, no calculators, no cell phones or electronic devices allowed but you may keep your pet blobfish for moral support.

All your answers must be fully justified, unless noted otherwise.

Exercise 1 (*3 points*). Let $\alpha \in \mathbb{R}_+^*$. Determine whether the following numerical series converges or diverges

$$\sum_n \left(\exp\left(\frac{(-1)^n}{n^\alpha}\right) - 1\right).$$

Exercise 2 (*5 points*). The goal of this exercise is to determine a numerical approximation of the sum of the series

$$S = \sum_{n=0}^{+\infty} \frac{n}{1+n^3}.$$

1. Briefly justify that the series $\sum_n n/(1+n^3)$ converges.

   For $N \in \mathbb{N}$ we define

$$S_N = \sum_{n=0}^{N} \frac{n}{1+n^3} \qquad \text{and} \qquad R_N = \sum_{n=N+1}^{+\infty} \frac{n}{1+n^3}.$$

2. Check that

$$\forall t \in (0,+\infty),\ \frac{t}{1+t^3} \le \frac{1}{t^2},$$
$$\forall N \in \mathbb{N}^*,\ \forall t \in [N+1,+\infty),\ \frac{1}{\left(1+\frac{1}{(N+1)^3}\right)t^2} \le \frac{t}{1+t^3}.$$

3. Use the integral comparison test (after justifying that it's valid to use it) to show that:

$$\forall N \ge 2,\ \frac{(N+1)^2}{(N+1)^3+1} \le S - S_N \le \frac{1}{N}.$$

4. You're given:

$$S_{38} + \frac{39^2}{39^3+1} = 1.11131207930173167232971234871 64\ldots \quad \text{and} \quad S_{38} + \frac{1}{38} = 1.1119872753836424506240658994848\ldots$$

   Deduce an approximation of S correct to as many decimal places as you can.

Exercise 3 (*3 points*). The two questions of this exercise are independent from each other.

1. You are given that

$$\lim_{n\to+\infty} \left(1+\frac{1}{n}\right)^n = \mathrm{e}.$$

   Determine the radius of convergence of the power series

$$f(z) = \sum_{n=0}^{+\infty} \frac{n^n}{n!} z^n.$$

2. Let $\sum_n a_n x^n$ be a power series of radius of convergence $R > 0$, and let

$$\begin{array}{rccc} f : & (-R,R) & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \displaystyle\sum_{n=0}^{+\infty} a_n x^n \end{array}$$

be the associated function.

a) What is the radius of convergence R_g of the power series

$$\sum_n n a_n x^{2n+1}?$$

b) Determine an explicit expression of the function

$$\begin{array}{rccc} g : & (-R_g, R_g) & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \displaystyle\sum_{n=0}^{+\infty} n a_n x^{2n+1} \end{array}$$

in terms of f.

Exercise 4 (*8 points*)**.** The goal of this exercise is to find the solutions of the following differential equation

$$\text{(E)} \qquad x\big(x^2+1\big)y''(x) + \big(x^2-1\big)y'(x) = 1$$

that possess a power series expansion around 0.
Let f be a function defined by a power series of radius $R > 0$, say

$$\forall x \in (-R, R),\ f(x) = \sum_{n=0}^{+\infty} a_n x^n.$$

1. a) For $x \in (-R, R)$, express $x\big(x^2+1\big)f''(x) + \big(x^2-1\big)f'(x)$ as the sum of a power series.
   b) Show that f is a solution of Equation (E) on $(-R, R)$ if and only if

$$(*) \qquad \begin{cases} a_1 = -1 \\ \forall n \geq 2,\ (n-1)a_{n-1} + (n+1)a_{n+1} = 0. \end{cases}$$

2. From now on we assume that the coefficients $(a_n)_{n\in\mathbb{N}}$ satisfy the condition $(*)$.
   a) Deduce, in terms of a_2, an explicit expression of the coefficients a_n for $n \geq 1$. You will consider the odd terms and the even terms separately.
   b) Determine the radius of convergence R of the power series f. You will consider the odd and even components of f separately.
3. a) Recall (without any justifications) the power series expansion of the function

$$\begin{array}{rccc} h : & (-1,1) & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \dfrac{1}{1+x}. \end{array}$$

   b) Deduce the power series expansion of the expressions $F(x) = \ln(1+x)$ and $G(x) = \arctan(x)$, and specify their radii of convergence.
   c) Deduce an explicit expression of the solutions of Equation (E).

Exercise 5 (*1 points*)**.** The two questions of this exercise are independent from each other.

1. Let q be the quadratic form on $\mathbb{R}^3$ the matrix of which, in the standard basis of $\mathbb{R}^3$ is

$$[q]_{\text{std}} = M = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & -1 \end{pmatrix}.$$

   For $(x, y, z) \in \mathbb{R}^3$, give an explicit expression of $q(x, y, z)$. No justifications required.
2. Let q be the quadratic form on $\mathbb{R}^3$ defined by

$$\begin{array}{rccc} q : & \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ & (x,y,z) & \longmapsto & x^2 - y^2 - 3z^2 + 2xy + 3xz. \end{array}$$

   Give the matrix $M = [q]_{\text{std}}$ of q in the standard basis of $\mathbb{R}^3$. No justifications required.