INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Solution of Math Test #5

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Exercise 1.

1. The function f is *continuous* on the *closed* and *bounded* set C hence, by the Extreme Value Theorem, f is bounded and attains its bounds.

2. We determine the critical points of f in $\mathring{C}$: Let $(x,y)\in\mathring{C}$. Then:

$$\partial_1 f(x,y) = 4x^3-4, \qquad \partial_2 f(x,y) = 4y^3-\frac{1}{2},$$

hence

$$(x,y) \text{ is a critical point of } f \iff \begin{cases} 4x^3-4=0 \\ 4y^3-1/2=0 \end{cases} \iff (x,y)=(1,1/2).$$

We now study the Hessian matrix of f at this critical point: for $(x,y)\in C$,

$$\partial^2_{1,1}f(x,y)=12x^2, \qquad \partial^2_{1,2}f(x,y)=0, \qquad \partial^2_{2,2}f(x,y)=12y^2,$$

hence the Hessian matrix of f at $(1,1/2)$ is

$$H_{(1,1/2)}f = \begin{pmatrix} 12 & 0 \\ 0 & 3 \end{pmatrix}.$$

Its signature is $(2,0)$, hence f possesses a local minimum at $(1,1/2)$, and $f(1,1/2)=-51/16$.

We now study f on ∂C:

- On the lower horizontal side $[0,2]\times\{0\}$:

$$\forall x\in[0,2],\ \varphi(x)=f(x,0)=x^4-4x.$$

  We study the extreme values of φ:

$$\varphi'(x)=4x^3-4,$$

  hence φ possesses a unique critical point at 1, and $\varphi(1)=-3$. Moreover, $\varphi(0)=0$ and $\varphi(2)=8$, hence the minimum value of f on this side is -3 and its maximum is 8.

- On the upper horizontal side $[0,2]\times\{2\}$:

$$\forall x\in[0,2],\ \varphi(x)=f(x,2)=x^4+16-4x-1=x^4-4x+15.$$

  This is just a shifted version of the previous case, hence the minimum value of f on this side is 12 and its maximum is 23.

- On the left vertical side $\{0\}\times[0,2]$:

$$\forall y\in[0,2],\ \varphi(x)=f(0,y)=y^4-\frac{y}{2}.$$

  We study the extreme values of φ:

$$\varphi'(y)=4y^3-\frac{1}{2},$$

  hence φ possesses a unique critical points at $1/2$ and $\varphi(1/2)=-3/16$. Moreover, $\varphi(0)=0$ and $\varphi(2)=15$, hence the minimum value of f on this side is $-3/16$ and its maximum value is 15.

- On the right vertical side $\{2\}\times[0,2]$:

$$\forall y\in[0,2],\ \varphi(x)=f(2,y)=16+y^4-8-\frac{y}{2}=y^4-\frac{y}{2}+8.$$

  This is just a shifted version of the previous case, hence the minimum value of f on this side is $-3/16+8=125/16$ and its maximum value is 23.

Conclusion: the minimum value of f on C is $-51/16$ and is attained at $(1, 1/2)$ and the maximum value of f on C is 23 and is attained at $(2, 2)$.

Exercise 2.

1. 1 is an obvious eigenvalue of A since

$$\operatorname{rk}(A - I_3) = \operatorname{rk}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} = 1$$

and we conclude that its multiplicity is $3 - \operatorname{rk}(A - I_3) = 2$ (since A is diagonalizable since A is a real symmetric matrix). We use the trace to determine the other eigenvalue:

$$\operatorname{tr}(A) = 6 = 1 + 1 + \text{other eigenvalue}$$

hence 4 is the other eigenvalue of A.

We know that the eigenspaces of A are orthogonal; now the equation of the eigenspace E_1 associated with the eigenvalue 1 is

$$(E_1) \qquad x - y - z = 0$$

from which we read that $X_4 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \in E_1^{\perp}$, hence X_4 is an eigenvalue associated with the eigenvalue 4.

We now pick a random (non-nil) vector in E_1, say $X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. We use the cross-product to produce another vector in $E_4^{\perp} = E_1$, orthogonal to X_1:

$$Y_1 = X_4 \times X_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

At this point, we know that (X_1, Y_1, X_4) is an orthogonal family. We define the matrix

$$P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \end{pmatrix},$$

obtained by stacking the vectors X_1, Y_1, X_4 divided by their respective norm, so that P is an orthogonal matrix, and by setting

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

we have $A = P\,D\,{}^tP$.

Exercise 3.

1. We are given that φ is a symmetric bilinear form on E. Moreover,

   - φ is positive semi-definite: let $f \in E$. Then

   $$\varphi(f, f) = \int_0^1 f(t)^2 t \,\mathrm{d}t,$$

   and since $t \mapsto f(t)^2 t$ is non-negative on $[0, 1]$, we conclude that $\varphi(f, f) \geq 0$.
   - φ is positive definite: let $f \in E$ such that $\varphi(f, f) = 0$. Then, since $t \mapsto f(t)^2 t$ is *continuous* and *non-negative*, we conclude that

   $$\forall t \in [0, 1],\ f(t)^2 t = 0,$$

   hence

   $$\forall t \in (0, 1],\ f(t) = 0,$$

   and since f is continuous at 0, we conclude that $f(0) = 0$. Hence $\forall t \in [0, 1],\ f(t) = 0$, i.e., $f = 0_E$.

2. Let $k, \ell \in \mathbb{N}$. Then

$$\varphi(u_k, u_\ell) = \int_0^1 t^{k+\ell+1}\,\mathrm{d}t = \frac{1}{k+\ell+2}.$$

3.
   - We set $v_0 = u_0$.
   - We set $v_1 = u_1 + \lambda u_0$ for some $\lambda \in \mathbb{R}$. Then

$$v_1 \perp_\varphi v_0 \iff \varphi(v_1, v_0) = 0 \iff \varphi(u_1, u_0) + \lambda\varphi(u_0, u_0) = 0 \iff \lambda = -\frac{\varphi(u_1, u_0)}{\varphi(u_0, u_0)} = -\frac{1/3}{1/2} = -\frac{2}{3}.$$

   We hence set

$$v_1 = u_1 - \frac{2}{3}u_0.$$

   The family $\mathscr{B}_1' = (v_0, v_1)$ is an orthogonal basis of F_1.

4. Let $k \in \mathbb{N}$. Since $\mathscr{B}_1'$ is an orthogonal basis of F_1, we have

$$p_1(u_3) = \frac{\varphi(u_3, v_0)}{\varphi(v_0, v_0)}v_0 + \frac{\varphi(u_3, v_1)}{\varphi(v_1, v_1)}v_1.$$

   Now,

$$\varphi(u_3, v_1) = \varphi(u_3, u_1) - \frac{2}{3}\varphi(u_3, u_0) = \frac{1}{6} - \frac{2}{3}\frac{1}{5} = \frac{1}{30}$$
$$\varphi(v_1, v_1) = \varphi(u_1, u_1) - \frac{4}{3}\varphi(u_0, u_1) + \frac{4}{9}\varphi(u_0, u_0) = \frac{1}{4} - \frac{4}{3} \times \frac{1}{3} + \frac{4}{9} \times \frac{1}{2} = \frac{1}{36}.$$

   Hence

$$\begin{aligned} p_1(u_3) &= \frac{1/5}{1/2}v_0 + \frac{1/30}{1/36}v_1 \\ &= \frac{2}{5}v_0 + \frac{6}{5}v_1 \\ &= \frac{2}{5}u_0 + \frac{6}{5}\left(u_1 - \frac{2}{3}u_0\right) \\ &= -\frac{2}{5}u_0 + \frac{6}{5}u_1. \end{aligned}$$

5.

$$\begin{aligned} \min_{(a,b)\in\mathbb{R}^2} \int_0^1 \left(t^3 - at - b\right)^2 t\,\mathrm{d}t &= \min_{(a,b)\in\mathbb{R}^2} \|u_3 - au_1 - u_0\|_\varphi^2 \\ &= \min_{v\in F_1} \|u_3 - v\|_\varphi^2 \\ &= \|u_3 - p_1(u_3)\|_\varphi^2 \qquad \textit{by the property of the orthogonal projection} \\ &= \|u_3\|_\varphi^2 - \|p_1(u_3)\|_\varphi^2 \qquad \textit{by the Pythagorean Theorem} \\ &= \frac{1}{8} - \left\|\frac{2}{5}v_0 + \frac{6}{5}v_1\right\|_\varphi^2 \\ &= \frac{1}{8} - \left\|\frac{2}{5}v_0\right\|_\varphi^2 - \left\|\frac{6}{5}v_1\right\|_\varphi^2 \qquad \textit{by the Pythagorean Theorem} \\ &= \frac{1}{8} - \frac{4}{25} \times \frac{1}{2} - \frac{36}{25} \times \frac{1}{36} \\ &= \frac{1}{200}. \end{aligned}$$

Exercise 4.

Part I

1. Let $u_0, h \in E$. Recall that

$$q(u_0 + h) = q(u_0) + 2\varphi(u_0, h) + q(h)$$

and that, since β is linear, $\beta(u_0+h)=\beta(u_0)+\beta(h)$. Hence

$$f(u_0+h)=q(u_0+h)+\beta(h)=q(u_0)+2\varphi(u_0,h)+q(h)+\beta(u_0)+\beta(h),$$

as required.

2. Since φ is a bilinear form, φ is linear with respect to its second argument. We hence recognize μ_{u_0} as a linear combination of two linear maps; hence μ_{u_0} is linear. Now let $h\in E$ and denote by $H=[h]_{\mathrm{std}}$ its coordinates in the standard basis std of E. Then

$$\mu_{u_0}(h)=2\varphi(u_0,h)+\beta(h)=2\,{}^tU_0AH+BH=\big(2\,{}^tU_0A+B\big)H,$$

hence the matrix of μ_{u_0} in the standard basis std of E is

$$M_{u_0}=[\mu_{u_0}]_{\mathrm{std}}=2\,{}^tU_0A+B.$$

3. a) Let $u_0\in E$. Then

$$\begin{aligned}
\mu_{u_0}=\mathbf{0} &\iff M_{u_0}=\mathbf{0} \\
&\iff 2\,{}^tU_0A+B=\mathbf{0} && \textit{by Question 2}\\
&\iff {}^tU_0A=-\frac{1}{2}B \\
&\iff {}^tAU_0=-\frac{1}{2}\,{}^tB && \textit{since } {}^t\big({}^tU_0A\big)={}^tAU_0\\
&\iff AU_0=-\frac{1}{2}\,{}^tB && \textit{since } A \textit{ is symmetric}\\
&\iff U_0=-\frac{1}{2}A^{-1}\,{}^tB && \textit{since } A \textit{ is invertible}
\end{aligned}$$

Hence there exists a unique $u_0\in E$ such that $\mu_{u_0}=\mathbf{0}$, namely the element $u_0\in E$ such that

$$U_0=[u_0]_{\mathrm{std}}=-\frac{1}{2}A^{-1}\,{}^tB.$$

b) Let $h\in E$. Then, by Question 1,

$$f(u_0+h)=q(h)+2\varphi(u_0,h)+\beta(h)+q(u_0)+\beta(u_0)=q(h)+\mu_{u_0}(h)+q(u_0)+\beta(u_0).$$

Now, u_0 is such that $\mu_{u_0}(h)=0$, so that

$$f(u_0+h)=q(h)+q(u_0)+\beta(u_0).$$

Also, $0=\mu_{u_0}(u_0)=2q(u_0)+\beta(u_0)$, hence $\beta(u_0)=-2q(u_0)$, hence

$$f(u_0+h)=q(h)+q(u_0)-2q(u_0)=q(h)-q(u_0).$$

Part II

1. We set

$$q:\ \begin{array}{ccc}\mathbb{R}^2 & \longrightarrow & \mathbb{R}\\ (x,y) & \longmapsto & 13x^2+10xy+13y^2\end{array} \qquad\text{and}\qquad \beta:\ \begin{array}{ccc}\mathbb{R}^2 & \longrightarrow & \mathbb{R}\\ (x,y) & \longmapsto & 26\sqrt{2}x+10\sqrt{2}y.\end{array}$$

Clearly q is a quadratic form (since q is a homogeneous polynomial of degree 2) and β is linear. It is also clear that

$$\forall u\in\mathbb{R}^2,\ f(u)=q(u)+\beta(u).$$

2. We first observe that $A=[q]_{\mathrm{std}}=\begin{pmatrix}13&5\\5&13\end{pmatrix}$ satisfies $\det A=169-25=144\neq 0$, hence A is invertible, hence we can apply the result of Question 3 in Part I.

We know that $u_0=(x_0,y_0)$ satisfies

$$\begin{pmatrix}x_0\\y_0\end{pmatrix}=-\frac{1}{2}A^{-1}\,{}^tB$$

where

$$A = \begin{pmatrix} 13 & 5 \\ 5 & 13 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 26\sqrt{2} & 10\sqrt{2} \end{pmatrix}.$$

Now,

$$A^{-1} = \frac{1}{144}\begin{pmatrix} 13 & -5 \\ -5 & 13 \end{pmatrix}$$

hence

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = -\frac{1}{2 \times 144}\begin{pmatrix} 13 & -5 \\ -5 & 13 \end{pmatrix}\begin{pmatrix} 26\sqrt{2} \\ 10\sqrt{2} \end{pmatrix} = -\frac{\sqrt{2}}{144}\begin{pmatrix} 13 & -5 \\ -5 & 13 \end{pmatrix}\begin{pmatrix} 13 \\ 5 \end{pmatrix} = -\frac{\sqrt{2}}{144}\begin{pmatrix} 144 \\ 0 \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ 0 \end{pmatrix}.$$

Hence $u_0 = (-\sqrt{2}, 0)$.

3. 8 and 18 are eigenvalues of A since

$$A - 8I_3 = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix} \quad \text{and} \quad A - 18I_3 = \begin{pmatrix} -5 & 5 \\ 5 & -5 \end{pmatrix}$$

and these matrices are not invertible. It is now clear that

$$X_8 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \; X_{18} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

are eigenvectors of A associated with the eigenvalues 8 and 18 respectively. We set $v_1 = (1/\sqrt{2}, -1/\sqrt{2})$ and $v_2 = (1/\sqrt{2}, 1/\sqrt{2})$. It is clear that $\mathscr{B}' = (v_1, v_2)$ is an orthonormal (with respect to the standard dot product of $\mathbb{R}^2$) basis of $\mathbb{R}^2$ and that

$$[q]_{\mathscr{B}'} = A' = \begin{pmatrix} 8 & 0 \\ 0 & 18 \end{pmatrix}.$$

4. Let $[u_0]_{\mathscr{B}'} = \begin{pmatrix} x'_0 \\ y'_0 \end{pmatrix}$ be the coordinates of u_0 in the basis $\mathscr{B}'$. Then, for $u = (x, y) \in \mathbb{R}^2$ with coordinates $[u]_{\mathscr{B}'} = \begin{pmatrix} x' \\ y' \end{pmatrix}$ we obtain, from Question 3b of Part I:

$$f(u) = q(u - u_0) - q(u_0) = 8(x' - x'_0)^2 + 18(y' - y'_0)^2 - q(u_0).$$

Now, $q(u_0) = 13 \times (-\sqrt{2})^2 + 10 \times (-\sqrt{2}) \times 0 + 13 \times 0^2 = 26$, hence

$$f(u) = 8(x' - x'_0)^2 + 18(y' - y'_0)^2 - 26.$$

Hence

$$u \in (C) \iff 8(x'-x'_0)^2 + 18(y'-y'_0)^2 - 26 = 46 \iff 8(x'-x'_0)^2 + 18(y'-y'_0)^2 = 72 \iff \frac{(x' - x'_0)^2}{9} + \frac{(y' - y'_0)^2}{4} = 1$$

as required.

5. See Figure 8.

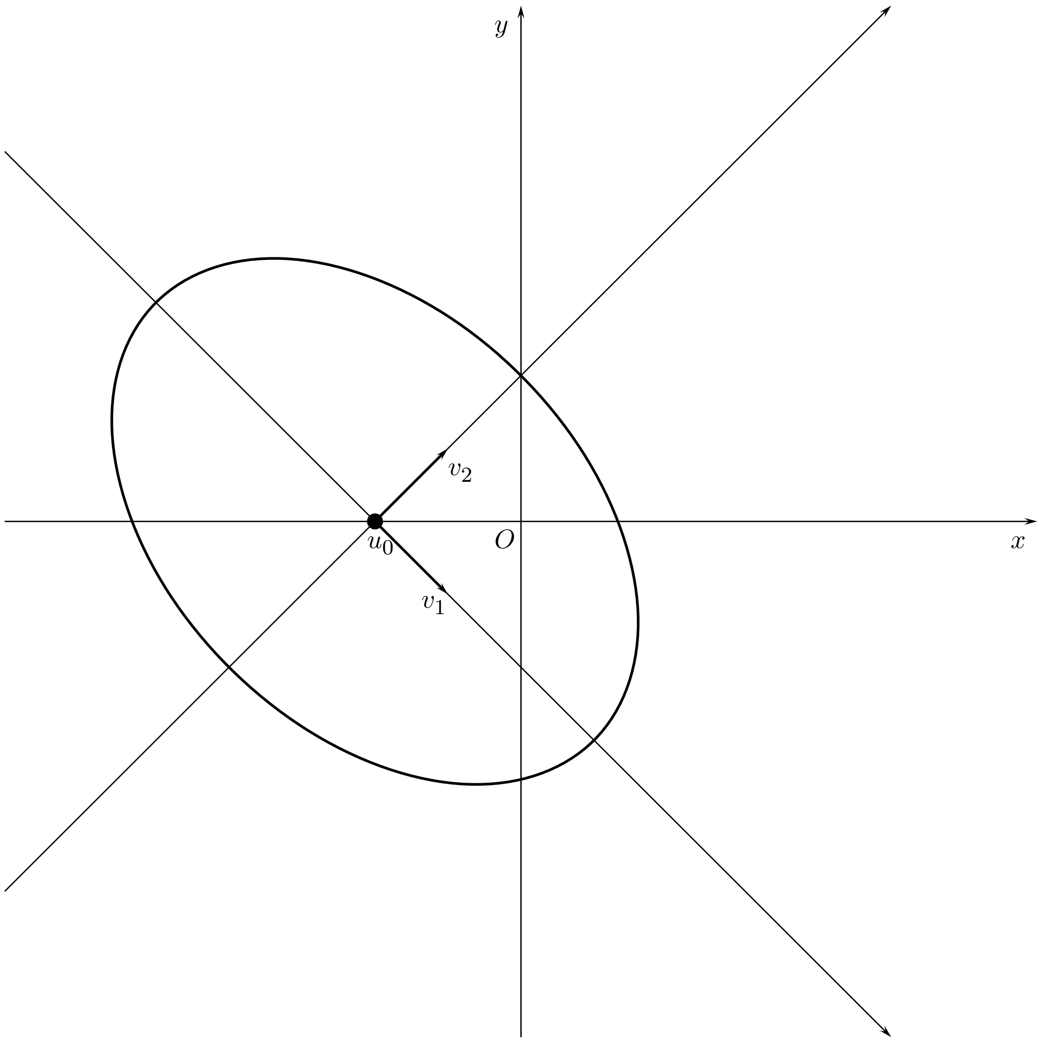

Figure 8. Ellipse (C) of Exercise 4