INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — S4— Math Test #5 — 2h

June 15, 2017

No documents, no calculators, no cell phones or electronic devices allowed but you may keep your pet blobfish for moral support.

All your answers must be fully justified, unless noted otherwise.

Exercise 1. Let $C = [0,2] \times [0,2]$. We define the function f as

$$f : \begin{array}{ccc} C & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & x^4 + y^4 - 4x - \dfrac{y}{2}. \end{array}$$

1. Explain why f possesses a global minimum and a global maximum.
2. Determine the value of $\min_C f$ and of $\max_C f$.

Exercise 2. Let

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix},$$

and let q be the quadratic form on $\mathbb{R}^3$ such that $[q]_{\text{std}} = A$, and let φ be the polar form of q.

1. Find an orthogonal matrix P and a diagonal matrix D such that $A = P\,D\,{}^tP$.
2. Is φ an inner product on $\mathbb{R}^3$?

Exercise 3. Let $E = C\big([0,1]\big)$ be the real vector space that consists of all real-valued continuous functions on $[0,1]$. We define the symmetric bilinear form φ on E as:

$$\varphi : \begin{array}{ccc} E \times E & \longrightarrow & \mathbb{R} \\ (f,g) & \longmapsto & \displaystyle\int_0^1 f(\mathbf{t})\, g(\mathbf{t})\, \mathbf{t}\, \mathrm{d}\mathbf{t} \end{array}$$

(mind the lonely $\mathbf{t}$ in the integral). For $k \in \mathbb{N}$ we define the element $u_k \in E$ as:

$$u_k : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{R} \\ t & \longmapsto & t^k. \end{array}$$

We define

$$F_1 = \operatorname{Span}\{u_0, u_1\},$$

so that F_1 consists of all polynomial functions on $[0,1]$ of degree non-greater than 1. You're given that $\mathscr{B}_1 = (u_0, u_1)$ is a basis of F_1 (i.e., that the vectors u_0 and u_1 are independent).

1. Show that φ is an inner product on E.
2. Let $k, \ell \in \mathbb{N}$. Compute the value of $\varphi(u_k, u_\ell)$.
3. Use the Gram-Schmidt process to obtain, from the basis $\mathscr{B}_1 = (u_0, u_1)$, an orthogonal (with respect to φ) basis $\mathscr{B}_1' = (v_0, v_1)$ of F_1.
4. Let $p_1 : E \to F_1$ be the orthogonal (with respect to φ) projection onto F_1. Determine, $p_1(u_3)$.
5. Deduce the value of

$$m = \min_{(a,b)\in\mathbb{R}^2} \int_0^1 \left(\mathbf{t}^3 - a\mathbf{t} - b\right)^2 \mathbf{t}\, \mathrm{d}\mathbf{t}.$$

Exercise 4. The two parts of this exercise are not independent: Part II uses the result of Part I.

Part I

Let $n \geq 2$ and let $E = \mathbb{R}^n$.

Let q be a quadratic form on E and let $\varphi : E \times E \to \mathbb{R}$ be its polar form. We denote by $A = [q]_{\text{std}} = [\varphi]_{\text{std}}$ the matrix of q (and hence of φ) in the standard basis std of E.

Let $\beta : E \to \mathbb{R}$ be a linear map. We denote by $B = [\beta]_{\text{std}}$ the matrix of β in the standard basis std of E.

We set

$$f : \begin{array}{rcl} E & \longrightarrow & \mathbb{R} \\ u & \longmapsto & q(u) + \beta(u). \end{array}$$

1. Show that

$$\forall u_0, h \in E, \qquad f(u_0 + h) \quad = \quad q(h) \quad + \quad 2\varphi(u_0, h) \ + \ \beta(h) \quad + \quad q(u_0) + \beta(u_0).$$

2. Let $u_0 \in E$. We denote by $U_0 = [u_0]_{\text{std}}$ its coordinates in the standard basis std of E. Briefly explain why the mapping

$$\mu_{u_0} : \begin{array}{rcl} E & \longrightarrow & \mathbb{R} \\ h & \longmapsto & 2\varphi(u_0, h) + \beta(h) \end{array}$$

is linear, and give its matrix $M_{u_0} = [\mu_{u_0}]_{\text{std}}$ in the standard basis std of E, in terms of $A = [\varphi]_{\text{std}}$, $B = [\beta]_{\text{std}}$ and $U_0 = [u_0]_{\text{std}}$.

3. In this question we assume that q is non-degenerate, i.e., that the matrix $A = [q]_{\text{std}}$ is invertible.

   a) Show that there exists a unique $u_0 \in E$ such that $\mu_{u_0} = \mathbf{0}$. Explicit the expression of U_0 in terms of A and B.

   b) Deduce that

$$\forall h \in E, \ f(u_0 + h) = q(h) - q(u_0),$$

   where u_0 is the element obtained in Question 3a. *Hint: you may find useful to use the fact that* $\mu_{u_0}(u_0) = 0$.

Part II

Let

$$f : \begin{array}{rcl} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x, y) & \longmapsto & 13x^2 + 10xy + 13y^2 + 26\sqrt{2}x + 10\sqrt{2}y. \end{array}$$

Let $(\mathscr{C})$ be the curve of $\mathbb{R}^2$ defined by

$$(\mathscr{C}) \qquad f(x, y) = 46.$$

1. Explicit the quadratic form q on $\mathbb{R}^2$ and the linear form β on $\mathbb{R}^2$ such that

$$\forall u \in \mathbb{R}^2, \ f(u) = q(u) + \beta(u).$$

2. Find, using the results of Part I, the unique element $u_0 = (x_0, y_0) \in \mathbb{R}^2$ such that

$$\forall (x, y) \in \mathbb{R}^2, \ f(x, y) = q(x - x_0, y - y_0) - q(x_0, y_0).$$

3. Determine an orthonormal basis $\mathscr{B}' = (v_1, v_2)$ (with respect to the standard dot product of $\mathbb{R}^2$) such that the matrix of q in $\mathscr{B}'$ is

$$[q]_{\mathscr{B}'} = A' = \begin{pmatrix} 8 & 0 \\ 0 & 18 \end{pmatrix}.$$

4. Explain why the equation of $(\mathscr{C})$ in $\mathscr{B}'$ is

$$\frac{(x' - x'_0)^2}{3^2} + \frac{(y' - y'_0)^2}{2^2} = 1,$$

   where the coordinates of $u_0 = (x_0, y_0)$ in $\mathscr{B}'$ are $[u_0]_{\mathscr{B}'} = \begin{pmatrix} x'_0 \\ y'_0 \end{pmatrix}$.

5. Plot the curve $(\mathscr{C})$. *Hint: start by plotting the point* u_0 *and the axes corresponding to* $\mathscr{B}'$ *that pass through* u_0.