INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Solution of Math Test #1

Romaric Pujol, romaric.pujol@insa-lyon.fr

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Exercise 1.

1. First observe that since the improper integral I is convergent, we have:

$$\lim_{n\to+\infty}\int_0^{x_n} f(t)\,\mathrm{d}t = I \text{ and } \lim_{n\to+\infty}\int_0^{y_n} f(t)\,\mathrm{d}t = I.$$

Now, for $n\in\mathbb{N}$,

$$\int_{x_n}^{y_n} f(t)\,\mathrm{d}t = \int_0^{y_n} f(t)\,\mathrm{d}t - \int_0^{x_n} f(t)\,\mathrm{d}t,$$

hence

$$\lim_{n\to+\infty}\int_{x_n}^{y_n} f(t)\,\mathrm{d}t = I - I = 0.$$

2. We first show that the improper integral

$$J = \int_0^{+\infty} \mathrm{e}^{-t^2}\,\mathrm{d}t$$

is convergent: the function $t\mapsto \mathrm{e}^{-t^2}$ is continuous on $\mathbb{R}_+$, hence the improper integral J is improper at $+\infty$. Now, for $t\in[1,+\infty)$, $-t^2\leq -t$, hence $0\leq \mathrm{e}^{-t^2}\leq \mathrm{e}^{-t}$. We know that the improper integral

$$\int_1^{+\infty}\mathrm{e}^{-t}\,\mathrm{d}t$$

converges hence, by the comparison test, the improper integral J converges too. Since

$$\lim_{n\to+\infty} n = +\infty \text{ and } \lim_{n\to+\infty}\mathrm{e}^n = +\infty$$

we conclude, by Question 1, that $\ell = 0$.

3. Define the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ by:

$$\forall n\in\mathbb{N},\ x_n = 2n\pi, \qquad y_n = (2n+1)\pi.$$

Clearly,

$$\lim_{n\to+\infty} x_n = +\infty \text{ and } \lim_{n\to+\infty} y_n = +\infty.$$

Now, let $n\in\mathbb{N}$ and let $t\in[x_n,y_n]$. Since $t\geq 0$ and $\sin(t)\geq 0$ we have $\mathrm{e}^{t\sin(t)}\geq 1$. Hence,

$$\int_{x_n}^{y_n}\mathrm{e}^{t\sin(t)}\,\mathrm{d}t \geq \int_{x_n}^{y_n}\mathrm{d}t = \pi \underset{n\to+\infty}{\not\to} 0.$$

Hence, by (the contrapositive of) Question 1, the improper integral (K) diverges.

Exercise 2.

1. Let $\alpha\in\mathbb{R}_+^*$. One has:

$$\forall x\in[0,+\infty), \qquad f_\alpha(x)^2 = \frac{1}{(1+x)^{2\alpha}}$$

$$f(x)\sqrt{1+f'_\alpha(x)^2} = \frac{1}{(1+x)^\alpha}\sqrt{1+\frac{\alpha^2}{(1+x)^{2\alpha+2}}}$$

The functions f_α^2 and $f\sqrt{1+(f'_\alpha)^2}$ being continuous on $[0,+\infty)$, the improper integrals C_α and D_α are improper at $+\infty$. Now, by Riemann at $+\infty$, we know that C_α converges if and only if $2\alpha>1$, i.e., if and only if $\alpha>1/2$. Also,

$$f(x)\sqrt{1+f'_\alpha(x)^2} \underset{x\to+\infty}{\sim} \frac{1}{x^\alpha} > 0,$$

hence, by the equivalent test and Riemann's criterion at $+\infty$, D_α diverges if and only if $\alpha\leq 1$. We hence conclude that

$$C_\alpha \text{ converges and } D_\alpha \text{ diverges} \iff \alpha\in(1/2,1].$$

2. A trumpet can be filled with paint if it has a finite volume, and can be painted if it has a finite surface area. Taking the trumpet obtained by rotating the graph of f_α about the origin yields a paradoxical trumpet when $\alpha \in [1/2, 1)$, since its volume is $2\pi C_\alpha$ (which is finite), yet its surface area is:

$$\lim_{A\to+\infty} 2\pi \int_0^{2A} f_\alpha(x)\sqrt{1+f_\alpha'(x)^2}\,\mathrm{d}x = +\infty$$

(the limit is $+\infty$ since we obtain a divergent improper integral of a non-negative function).

Exercise 3.

1. Let $f \in E$. Since f is bounded, there exists $M \in \mathbb{R}$ such that

$$\forall x \in \mathbb{R}_+, \left|f(x)\right| \leq M$$

(in fact, we can choose $M = \|f\|_\infty$). Then,

$$\forall t \in \mathbb{R}_+,\ 0 \leq \left|f(t)\right|\mathrm{e}^{-t} \leq M\mathrm{e}^{-t}.$$

Now, we know that the improper integral

$$\int_0^{+\infty} \mathrm{e}^{-t}\,\mathrm{d}t$$

converges, hence, by the comparison test, the improper integral

$$\int_0^{+\infty} \left|f(t)\right|\mathrm{e}^{-t}\,\mathrm{d}t$$

converges too.

2.
- Let $f \in E$ such that $N(f) = 0$. Then, since the function $t \mapsto \left|f(t)\right|\mathrm{e}^{-t}$ is *continuous* and *non-negative*, we conclude that

$$\forall t \in \mathbb{R}_+,\ \left|f(t)\right|\mathrm{e}^{-t} = 0,$$

hence $f = 0_E$.
- Let $f, g \in E$. Then,

$$N(f+g) = \int_0^{+\infty} \left|f(t)+g(t)\right|\mathrm{e}^{-t}\,\mathrm{d}t.$$

By the triangle inequality (for the absolute value), we know that

$$\forall t \in \mathbb{R}_+,\ \left|f(t)+g(t)\right|\mathrm{e}^{-t} \leq \left|f(t)\right|\mathrm{e}^{-t} + \left|g(t)\right|\mathrm{e}^{-t},$$

hence (since all the integrals are convergent),

$$N(f+g) = \int_0^{+\infty} \left|f(t)+g(t)\right|\mathrm{e}^{-t}\,\mathrm{d}t \leq \int_0^{+\infty} \left|f(t)\right|\mathrm{e}^{-t}\,\mathrm{d}t + \int_0^{+\infty} \left|g(t)\right|\mathrm{e}^{-t}\,\mathrm{d}t = N(f) + N(g).$$

- Let $f \in E$ and $\lambda \in \mathbb{R}$. Then:

$$N(\lambda f) = \int_0^{+\infty} \left|\lambda f(t)\right|\mathrm{e}^{-t}\,\mathrm{d}t = |\lambda| \int_0^{+\infty} \left|f(t)\right|\mathrm{e}^{-t}\,\mathrm{d}t = |\lambda| N(f).$$

Hence N is a norm on E.

3. a) Let $n \in \mathbb{N}$. Then,

$$N(f_n - 0_E) = \int_0^{+\infty} f_n(t)\mathrm{e}^{-t}\,\mathrm{d}t = \int_0^{+\infty} \mathrm{e}^{-(n+1)t}\,\mathrm{d}t = \frac{1}{n+1} \underset{n\to+\infty}{\longrightarrow} 0.$$

Hence the sequence $(f_n)_{n\in\mathbb{N}}$ converges to 0_E for the norm N.

b) Let $n \in \mathbb{N}$. Then,

$$\|f_n - 0_E\|_\infty = \sup_{t\in\mathbb{R}_+} \left|f_n(t)\right| = \sup_{t\in\mathbb{R}_+} \mathrm{e}^{-(n+1)t} = 1 \underset{n\to+\infty}{\not\longrightarrow} 0,$$

hence the sequence $(f_n)_{n\in\mathbb{N}}$ doesn't converge to 0_E for the norm $\|\cdot\|_\infty$.

4. The two norms N and $\|\cdot\|_\infty$ are not equivalent for otherwise the sequence $(f_n)_{n\in\mathbb{N}}$ would have the same convergence for both norms, but this is not the case as shown by Questions 3a) and 3b).

Exercise 4.

1. Define the mapping

$$\begin{array}{rccc}\varphi : & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (x,y) & \longmapsto & (x-2y, x+y).\end{array}$$

Clearly, the mapping φ is an endomorphism of $\mathbb{R}^2$ (linearity is obvious). Now, the matrix of φ in the standard basis of $\mathbb{R}^2$ is

$$[\varphi]_{\text{std}} = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix},$$

the determinant of which is $\det[\varphi]_{\text{std}} = 3 \neq 0$. Hence φ is invertible. Since

$$\forall u \in \mathbb{R}^2,\ N(u) = \left\|\varphi(u)\right\|_\infty,$$

we conclude (φ being an invertible linear mapping) that N is a norm on $\mathbb{R}^2$.

2. Moreover, we know that closed the unit ball $\overline{B}$ of N is obtained from the closed unit ball $\overline{B_\infty}$ of $\|\cdot\|_\infty$ by applying φ^{-1}:

$$\overline{B} = \varphi{-}1\left(\overline{B_\infty}\right).$$

Now, the matrix of φ^{-1} in the standard basis of $\mathbb{R}^2$ is:

$$[\varphi^{-1}]_{\text{std}} = [\varphi_{\text{std}}]^{-1} = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}.$$

The ball $\overline{B_\infty}$ is the symmetric parallelogram with vertices $(1,1)$ and $(1,-1)$, hence the ball $\overline{B}$ is the symmetric parallelogram with vertices $(1,0)$ and $(-1/3,-2/3)$. See figure 1.

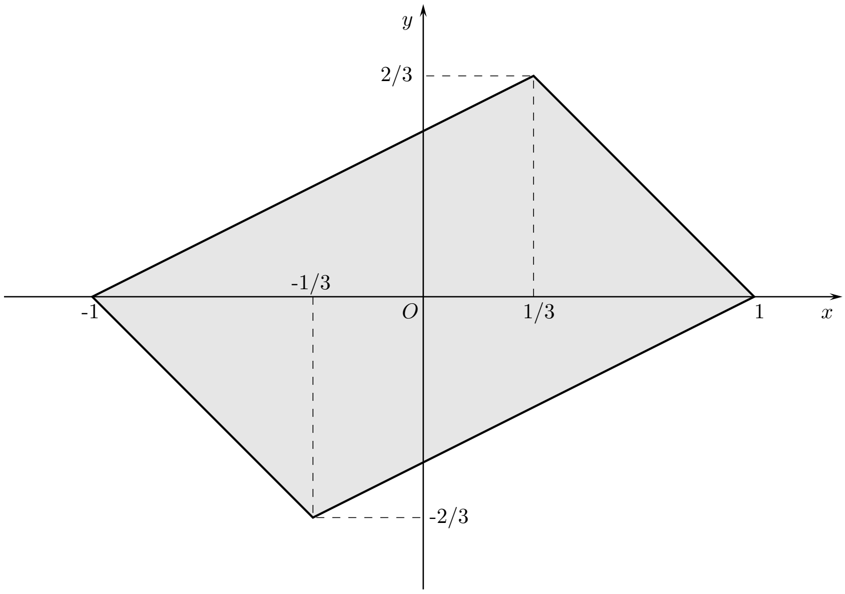

Figure 1. Closed unit ball $\overline{B}$ of N, of Exercise 4

Exercise 5.

1.
- Let $u \in E$ such that $N(u) = 0$. Then, $N_1(u) + N_2(u) = 0$, and since $N_1(u) \geq 0$ and $N_2(u) \geq 0$, we conclude that $N_1(u) = N_2(u) = 0$ hence, since N_1 is a norm, $u = 0_E$.
- Let $u, v \in E$. Then,

$$\begin{aligned} N(u+v) &= N_1(u+v) + N_2(u+v) \\ &\leq N_1(u) + N_1(v) + N_2(u) + N_2(v) \qquad \textit{by the triangle inequality of } N_1 \textit{ and } N_2 \\ &= N_1(u) + N_2(u)\ +\ N_1(v) + N_2(v) = N(u) + N(v). \end{aligned}$$

- Let $u \in E$ and $\lambda \in \mathbb{R}$. Then,
$$N(\lambda u) = N_1(\lambda u) + N_2(\lambda u) = |\lambda| N_1(u) + |\lambda| N_2(u) = |\lambda|\big(N_1(u) + N_2(u)\big) = |\lambda| N(u).$$

2. $\overline{B} \subset \overline{B_1}$. Indeed, let $u \in B$. This means that $N(u) \leq 1$, i.e., that $N_1(u) + N_2(u) \leq 1$. Since $N_2(u) \geq 0$, we conclude that $N_1(u) \leq 1$, hence $u \in \overline{B_1}$.

3. If the norms N_1 and N_2 are equivalent, there exists $\alpha, \beta \in \mathbb{R}_+^*$ such that
$$\alpha N_1 \leq N_2 \leq \beta N_1.$$
Hence, adding N_1 to all three terms of this inequality yields
$$N_1 + \alpha N_1 \leq N_1 + N_2 \leq N_1 + \beta N_1,$$
i.e.,
$$(\alpha + 1) N_1 \leq N \leq (\beta + 1) N_1,$$
and since $\alpha + 1 > 0$ and $\beta + 1 > 0$, we conclude that N and N_1 are equivalent.

Exercise 6. Let $(x, y) \in \mathbb{R}^2 \setminus \big\{(0,0)\big\}$. Then,
$$\big|f(x,y)\big| = \left|\frac{x|y|^{3/2}}{x^2 + y^2}\right| = \frac{|x||y|^{3/2}}{\big\|(x,y)\big\|_2^2} \leq \frac{\big\|(x,y)\big\|_2^{5/2}}{\big\|(x,y)\big\|_2^2} = \big\|(x,y)\big\|^{1/2} \underset{\|(x,y)\|_2 \to 0}{\longrightarrow} 0,$$
where we have used the useful inequalities
$$|x| \leq \big\|(x,y)\big\|_2 \text{ and } |y| \leq \big\|(x,y)\big\|_2.$$
Hence
$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

Exercise 7. Let $P \in E$, say $P = aX^2 + bX + c$. Then
$$\Phi(P) = (2aX + b)^2 = 4a^2X^2 + 4abX + b^2.$$
Now let $P_0 \in E$ and $H \in E$, say $P_0 = a_0X^2 + b_0X + c_0$ and $H = \alpha X^2 + \beta X + \gamma$. Then
$$\begin{aligned}\Phi(P_0 + H) &= 4(a_0 + \alpha)^2 X^2 + 4(a_0 + \alpha)(b_0 + \beta)X + (b_0 + \beta)^2 \\ &= \underbrace{4a_0^2X^2 + 4a_0b_0X + b_0^2}_{\Phi(P_0)} + \underbrace{8a_0\alpha X^2 + 4(a_0\beta + b_0\alpha)X + 2b_0\beta}_{\text{linear wrt } H} + \underbrace{4\alpha^2X^2 + 4\alpha\beta X + \beta^2}_{\text{remainder}}.\end{aligned}$$
We choose a norm on E (since E is a finite dimensional vector space, all norms are equivalent), let's call it N, defined by:
$$\forall a, b, c \in \mathbb{R},\ N\big(aX^2 + bX + c\big) = \sqrt{a^2 + b^2 + c^2}.$$
Clearly, N is a norm, as it is the 2-norm associated with the standard basis of E. For $H = \alpha X^2 + \beta X + \gamma \neq 0_E$,
$$\begin{aligned}\frac{N(\text{remainder})}{N(H)} &= \frac{\sqrt{16\alpha^4 + 16\alpha^2\beta^2 + \beta^4}}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}} \\ &\leq \frac{\sqrt{16\alpha^4 + 32\alpha^2\beta^2 + 16\beta^4}}{\sqrt{\alpha^2 + \beta^2}} \\ &= \frac{4\sqrt{\big(\alpha^2 + \beta^2\big)^2}}{\sqrt{\alpha^2 + \beta^2}} \\ &= 4\sqrt{\alpha^2 + \beta^2} \\ &\leq 4\sqrt{\alpha^2 + \beta^2 + \gamma^2} \\ &\leq 4N(H) \underset{H \to 0}{\longrightarrow} 0.\end{aligned}$$
Hence, Φ is differentiable at P_0 and
$$\begin{aligned} D_{P_0}H\ :\quad & E && \longrightarrow && E \\ & \alpha X^2 + \beta X + \gamma && \longmapsto && 8a_0\alpha X^2 + 4(a_0\beta + b_0\alpha)X + 2b_0\beta.\end{aligned}$$