

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. Let $f: \mathbb{R}_+ \to \mathbb{R}$ be a continuous function such that the following improper integral converges:

$$I=\int_0^{+\infty}f(t)\,\mathrm{d}t.$$

1. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two sequences of elements of \mathbb{R}_+ such that $\lim_{n\to+\infty}x_n=+\infty$ and $\lim_{n\to+\infty}y_n=+\infty$. Show that

$$\lim_{n\to+\infty}\int_{x_n}^{y_n}f(t)\,\mathrm{d}t=0.$$

2. Use the result of Question 1 to determine the value of the following limit:

$$\ell = \lim_{n \to +\infty} \int_{n}^{e^{n}} e^{-t^{2}} dt.$$

3. Use the result of Question 1 to show that the following improper integral diverges:

$$\int_0^{t+\infty} e^{t \sin(t)} dt.$$

Exercise 2. Let $f: \mathbb{R}_+ \to \mathbb{R}_+$ be a function of class C^1 . Let \mathscr{S} be the (infinite) surface obtained by rotating the graph of f about the horizontal axis. For $A \in \mathbb{R}_+^*$ we denote by \mathscr{S}_A the portion of \mathscr{S} enclosed between the planes x = 0 and x = A. We recall that the surface area of \mathscr{S}_A is

$$S_A = 2\pi \int_0^A f(x)\sqrt{1 + f'(x)^2} \, \mathrm{d}x,$$

and the volume enclosed by \mathcal{S}_A and the planes x = 0 and x = A is

$$V_A = \pi \int_0^A f(x)^2 \, \mathrm{d}x.$$

For $\alpha \in \mathbb{R}_+^*$ we define

$$f_{\alpha}: \mathbb{R}_{+} \longrightarrow \mathbb{R}_{+}$$

$$x \longmapsto \frac{1}{(1+x)^{\alpha}}.$$

1. Show that there exists $\alpha \in \mathbb{R}_+^*$ such that the improper integral

$$C_{\alpha} = \int_0^{+\infty} f_{\alpha}(x)^2 \, \mathrm{d}x$$

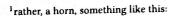
converges and (at the same time) the improper integral

$$D_{\alpha} = \int_0^{+\infty} f_{\alpha}(x) \sqrt{1 + f_{\alpha}'(x)^2} \, \mathrm{d}x$$

diverges, and determine all such α .

2. (Bonus) Explain why the following paradox can be stated:

There exists an infinite trumpet that can be filled with paint but that can't be painted.





Exercise 3. Let $E = C_b(\mathbb{R}_+, \mathbb{R})$ be the vector space that consists of all *continuous* and *bounded* real-valued functions defined on \mathbb{R}_+ . You don't need to justify that E is a vector space. For $f \in E$ we define:

$$||f||_{\infty} = \sup |f|.$$

You're given that $||f||_{\infty}$ is a norm on E.

1. Let $f \in E$. Show that the following improper integral

$$\int_0^{+\infty} |f(t)| \mathrm{e}^{-t} \, \mathrm{d}t$$

is convergent.

We hence define the mapping N as

$$N: E \longrightarrow \mathbb{R}_+$$
$$f \longmapsto \int_0^{+\infty} |f(t)| e^{-t} dt.$$

- 2. Show that N is a norm on E.
- 3. For $n \in \mathbb{N}$ we define

$$f_n: \mathbb{R}_+ \longrightarrow \mathbb{R}$$
$$t \longmapsto e^{-nt}.$$

Clearly, $f_n \in E$.

- a) Show that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges to 0_E for the norm N.
- b) Does the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converge to 0_E for the norm $\|\cdot\|_{\infty}$?
- 4. Are the norms N and $\|\cdot\|_{\infty}$ equivalent norms?

Exercise 4. Let

$$\begin{array}{ccc} N: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}_+ \\ & (x,y) & \longmapsto & \max \big\{ |x-2y|, |x+y| \big\}. \end{array}$$

- 1. Show that N is a norm on \mathbb{R}^2 .
- 2. Plot the closed unit ball \overline{B} of N.

Exercise 5. Let E be a vector space, and let N_1 and N_2 be two norms on E. We define $N = N_1 + N_2$.

- 1. Show that N is a norm on E.
- 2. What relation exists between the closed unit ball \overline{B} of N and the closed unit ball $\overline{B_1}$ of N_1 ?
- 3. Show that if the norms N_1 and N_2 are equivalent, then the norms N and N_1 are equivalent.

Exercise 6. We define the function f as

$$f: \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto \frac{x|y|^{3/2}}{x^2 + y^2}.$$

Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ exists and determine its value.

Exercise 7. Let $E = \mathbb{R}_2[X]$ be the vector space of all polynomials of degree at most 2 with real coefficients. Let

$$\Phi: E \longrightarrow E$$

$$P \longmapsto (P')^2.$$

You're given that Φ is well-defined. Let $P_0 \in E$. Show that Φ is differentiable at P_0 and determine $D_{P_0}\Phi$ explicitly.