INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — S1 — Math Test #1 — 2h

November 6, 2017

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function such that the following improper integral converges:

$$I = \int_0^{+\infty} f(t)\,\mathrm{d}t.$$

1. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two sequences of elements of $\mathbb{R}_+$ such that $\lim_{n\to+\infty} x_n = +\infty$ and $\lim_{n\to+\infty} y_n = +\infty$. Show that

$$\lim_{n\to+\infty} \int_{x_n}^{y_n} f(t)\,\mathrm{d}t = 0.$$

2. Use the result of Question 1 to determine the value of the following limit:

$$\ell = \lim_{n\to+\infty} \int_n^{\mathrm{e}^n} \mathrm{e}^{-t^2}\,\mathrm{d}t.$$

3. Use the result of Question 1 to show that the following improper integral diverges:

$$\text{(K)} \qquad \int_0^{+\infty} \mathrm{e}^{t\sin(t)}\,\mathrm{d}t.$$

Exercise 2. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a function of class C^1. Let $\mathscr{S}$ be the (infinite) surface obtained by rotating the graph of f about the horizontal axis. For $A \in \mathbb{R}_+^*$ we denote by $\mathscr{S}_A$ the portion of $\mathscr{S}$ enclosed between the planes $x = 0$ and $x = A$. We recall that the surface area of $\mathscr{S}_A$ is

$$S_A = 2\pi \int_0^A f(x)\sqrt{1 + f'(x)^2}\,\mathrm{d}x,$$

and the volume enclosed by $\mathscr{S}_A$ and the planes $x = 0$ and $x = A$ is

$$V_A = \pi \int_0^A f(x)^2\,\mathrm{d}x.$$

For $\alpha \in \mathbb{R}_+^*$ we define

$$f_\alpha : \begin{array}{rcl} \mathbb{R}_+ & \longrightarrow & \mathbb{R}_+ \\ x & \longmapsto & \dfrac{1}{(1+x)^\alpha} \end{array}.$$

1. Show that there exists $\alpha \in \mathbb{R}_+^*$ such that the improper integral

$$C_\alpha = \int_0^{+\infty} f_\alpha(x)^2\,\mathrm{d}x$$

converges and (at the same time) the improper integral

$$D_\alpha = \int_0^{+\infty} f_\alpha(x)\sqrt{1 + f_\alpha'(x)^2}\,\mathrm{d}x$$

diverges, and determine all such α.

2. *(Bonus)* Explain why the following paradox can be stated:

There exists an infinite trumpet[1] that can be filled with paint but that can't be painted.

[1] rather, a horn, something like this: 

Exercise 3. Let $E = C_b(\mathbb{R}_+, \mathbb{R})$ be the vector space that consists of all *continuous* and *bounded* real-valued functions defined on $\mathbb{R}_+$. You don't need to justify that E is a vector space. For $f \in E$ we define:

$$\|f\|_\infty = \sup|f|.$$

You're given that $\|f\|_\infty$ is a norm on E.

1. Let $f \in E$. Show that the following improper integral

$$\int_0^{+\infty} |f(t)| e^{-t}\, dt$$

is convergent.

We hence define the mapping N as

$$N : \begin{array}{ccc} E & \longrightarrow & \mathbb{R}_+ \\ f & \longmapsto & \displaystyle\int_0^{+\infty} |f(t)| e^{-t}\, dt. \end{array}$$

2. Show that N is a norm on E.

3. For $n \in \mathbb{N}$ we define

$$f_n : \begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \mathbb{R} \\ t & \longmapsto & e^{-nt}. \end{array}$$

Clearly, $f_n \in E$.

a) Show that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges to 0_E for the norm N.

b) Does the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converge to 0_E for the norm $\|\cdot\|_\infty$?

4. Are the norms N and $\|\cdot\|_\infty$ equivalent norms?

Exercise 4. Let

$$N : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}_+ \\ (x, y) & \longmapsto & \max\{|x - 2y|, |x + y|\}. \end{array}$$

1. Show that N is a norm on $\mathbb{R}^2$.

2. Plot the closed unit ball $\overline{B}$ of N.

Exercise 5. Let E be a vector space, and let N_1 and N_2 be two norms on E. We define $N = N_1 + N_2$.

1. Show that N is a norm on E.

2. What relation exists between the closed unit ball $\overline{B}$ of N and the closed unit ball $\overline{B_1}$ of N_1?

3. Show that if the norms N_1 and N_2 are equivalent, then the norms N and N_1 are equivalent.

Exercise 6. We define the function f as

$$f : \begin{array}{ccc} \mathbb{R}^2 \setminus \{(0,0)\} & \longrightarrow & \mathbb{R} \\ (x, y) & \longmapsto & \dfrac{x|y|^{3/2}}{x^2 + y^2}. \end{array}$$

Show that $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$ exists and determine its value.

Exercise 7. Let $E = \mathbb{R}_2[X]$ be the vector space of all polynomials of degree at most 2 with real coefficients. Let

$$\Phi : \begin{array}{ccc} E & \longrightarrow & E \\ P & \longmapsto & (P')^2. \end{array}$$

You're given that Φ is well-defined. Let $P_0 \in E$. Show that Φ is differentiable at P_0 and determine $D_{P_0}\Phi$ explicitly.