SCAN 2 — Solution of Math Test #2

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Exercise 1.

1. a) For $(u,v)\in D$, $\mathrm{e}^{u+v}\in\mathbb{R}_+^*$, and since $u-v>0$, $\ln(u-v)$ is well-defined and belongs to $\mathbb{R}$. Hence $\bigl(\mathrm{e}^{u+v},\ln(u-v)\bigr)$ is well-defined and belongs to Ω.

   b) • Clearly φ is of class C^1 (and even of class C^∞).

   • We now show that φ is a bijection (and we'll also show have an explicit expression for φ^{-1}): let $(u,v)\in D$ and $(x,y)\in\Omega$. Then

$$\varphi(u,v)=(x,y)\iff\begin{cases}\mathrm{e}^{u+v}=x\\ \ln(u-v)=y\end{cases}\iff\begin{cases}u+v=\ln(x)\\ u-v=\mathrm{e}^y\end{cases}\iff\begin{cases}u=\dfrac{\ln(x)+\mathrm{e}^y}{2}\\ v=\dfrac{\ln(x)-\mathrm{e}^y}{2}.\end{cases}$$

   Hence φ is a bijection and

$$\begin{aligned}\varphi^{-1}\ :\quad \Omega &\longrightarrow D\\ (x,y)&\longmapsto\left(\frac{\ln(x)+\mathrm{e}^y}{2},\frac{\ln(x)-\mathrm{e}^y}{2}\right).\end{aligned}$$

   • Clearly, from the form of φ^{-1} we notice that φ^{-1} is of class C^1 (and even of class C^∞).[1]

   Hence φ is a C^1-diffeomorphism.

   c) i) Let $(u,v)\in D$. Then

$$J_{(u,v)}\varphi=\begin{pmatrix}\mathrm{e}^{u+v}&\mathrm{e}^{u+v}\\ 1/(u-v)&-1/(u-v)\end{pmatrix}.$$

   ii) Let $(x,y)\in\Omega$. Then

$$J_{(x,y)}(\varphi^{-1})=\begin{pmatrix}1/2x&\mathrm{e}^y/2\\ 1/2x&-\mathrm{e}^y/2\end{pmatrix}.$$

   iii) The relation between $J\varphi$ and $J\bigl(\varphi^{-1}\bigr)$ is:

$$\forall(u,v)\in D,\ \bigl(J_{(u,v)}\varphi\bigr)^{-1}=J_{\varphi(u,v)}\bigl(\varphi^{-1}\bigr)$$

   or, equivalently,

$$\forall(x,y)\in\Omega,\ \bigl(J_{\varphi^{-1}(x,y)}\varphi\bigr)^{-1}=J_{(x,y)}\bigl(\varphi^{-1}\bigr)$$

   We show the first relation: For $(u,v)\in D$,

$$\bigl(J_{(u,v)}\varphi\bigr)^{-1}=-\frac{u-v}{2\mathrm{e}^{u+v}}\begin{pmatrix}-1/(u-v)&-\mathrm{e}^{u+v}\\ -1/(u-v)&\mathrm{e}^{u+v}\end{pmatrix}=\begin{pmatrix}1/2\mathrm{e}^{u+v}&(u-v)/2\\ 1/2\mathrm{e}^{u+v}&-(u-v)/2\end{pmatrix},$$

   and

$$J_{\varphi(u,v)}\bigl(\varphi^{-1}\bigr)=J_{(\mathrm{e}^{u+v},\ln(u-v))}\bigl(\varphi^{-1}\bigr)=\begin{pmatrix}1/2\mathrm{e}^{u+v}&(u-v)/2\\ 1/2\mathrm{e}^{u+v}&-(u-v)/2\end{pmatrix}.$$

2. Let $f:\Omega\to\mathbb{R}$ and define $g:D\to\mathbb{R}$ by $g=f\circ\varphi$. Since φ is a C^1-diffeomorphism, we know that f is of class C^1 if and only if g is of class C^1. We assume that f is of class C^1.

   Observe that we have:

$$\forall(u,v)\in D,\ g(u,v)=f\bigl(\mathrm{e}^{u+v},\ln(u-v)\bigr),$$

   hence, for $(u,v)\in D$, set $(x,y)=\varphi(u,v)$, and it follows from the Chain Rule that

$$\partial_2 g(u,v)=\mathrm{e}^{u+v}\partial_1 f(x,y)-1/(u-v)\partial_2 f(x,y)=x\partial_1 f(x,y)-\mathrm{e}^{-y}\partial_2 f(x,y).$$

   Hence

$$\begin{aligned}f\text{ is a solution of }(*)&\iff\forall(x,y)\in\Omega,\ x\partial_1 f(x,y)-\mathrm{e}^{-y}\partial_2 f(x,y)-3f(x,y)=0\\ &\iff\forall(u,v)\in D,\ \partial_2 g(u,v)-3g(u,v)=0\\ &\iff\exists A:\mathbb{R}\to\mathbb{R}\text{ of class }C^1,\ \forall(u,v)\in D,\ g(u,v)=A(u)\mathrm{e}^{3v}\\ &\iff\exists A:\mathbb{R}\to\mathbb{R}\text{ of class }C^1,\ \forall(x,y)\in\Omega,\ f(x,y)=A\bigl(\ln(x)+\mathrm{e}^y\bigr)\mathrm{e}^{3\left(\ln(x)-\mathrm{e}^y\right)/2}.\end{aligned}$$

[1] We could also have computed the Jacobian matrix of φ and used the GIFT, but this is asked in the next question.

Exercise 2.

1. Since U is simply-connected, we only need to show that ω is closed. Notice that since f is of class C^2, ω is of class C^1. Denoting by P_1 and P_2 the components of ω, namely, $P_1 = -\partial_2 f$ and $P_2 = \partial_1 f$ we have:

$$\partial_2 P_1 = -\partial^2_{2,2} f = \partial^2_{1,1} f = \partial_1 P_2$$

where we used the fact that f is harmonic in the middle equality. Hence ω is closed; moreover, since ω is of class C^1, by Poincaré's Lemma, ω is exact and there exists a function $g : U \to \mathbb{R}$ of class C^2 such that $\mathrm{d}g = \omega$. Of course, g can be chosen such that $g(0,0) = 0$ (for otherwise, take $g - g(0,0)$ instead).

2. Notice that $\partial_1 g = -\partial_2 f$ and $\partial_2 g = \partial_1 f$. Hence,

$$\partial^2_{1,1} g + \partial^2_{2,2} g = -\partial^2_{1,2} f + \partial^2_{2,1} f = 0,$$

by Schwarz' Lemma, since f is of class C^2. Hence g is harmonic.

3. a) Notice that:

$$\begin{aligned} &f(0,0) = g(0,0) = 0, && \partial_1 g(0,0) = -\partial_2 f(0,0), && \partial_2 g(0,0) = \partial_1 f(0,0),\\ &\partial^2_{1,1} g(0,0) = -\partial^2_{1,2} f(0,0), && \partial^2_{1,2} g(0,0) = -\partial^2_{2,2} f(0,0) = \partial^2_{1,1} f(0,0), && \partial^2_{2,2} g(0,0) = \partial^2_{1,2} f(0,0), \end{aligned}$$

We'll use these equalities in the sequel. By the second-order Taylor–Young expansion of f and g at $(0,0)$ we have (using $f(0,0) = g(0,0) = 0$):

$$\begin{aligned} xf(x,y) + yg(x,y) \underset{(x,y)\to(0,0)}{=} \; & x\Big(x\partial_1 f(0,0) + y\partial_2 f(0,0) \\ & \quad + \frac{1}{2}\big(x^2\partial^2_{1,1} f(0,0) + 2xy\partial^2_{1,2} f(0,0) + y^2\partial^2_{2,2} f(0,0)\big) + o\big(x^2+y^2\big)\Big) \\ & + y\Big(x\partial_1 g(0,0) + y\partial_2 g(0,0) \\ & \quad + \frac{1}{2}\big(x^2\partial^2_{1,1} g(0,0) + 2xy\partial^2_{1,2} g(0,0) + y^2\partial^2_{2,2} g(0,0)\big) + o\big(x^2+y^2\big)\Big) \\ \underset{(x,y)\to(0,0)}{=} \; & x^2\partial_1 f(0,0) + xy\partial_2 f(0,0) + \frac{1}{2}\big(x^3\partial^2_{1,1} f(0,0) + 2x^2y\partial^2_{1,2} f(0,0) + xy^2\partial^2_{2,2} f(0,0)\big) \\ & + xy\partial_1 g(0,0) + y^2\partial_2 g(0,0) + \frac{1}{2}\big(x^2y\partial^2_{1,1} g(0,0) + 2xy^2\partial^2_{1,2} g(0,0) + y^3\partial^2_{2,2} g(0,0)\big) \\ & + o\left(\big(x^2+y^2\big)^{3/2}\right) \\ \underset{(x,y)\to(0,0)}{=} \; & x^2\partial_1 f(0,0) + xy\partial_2 f(0,0) + \frac{1}{2}\big(x^3\partial^2_{1,1} f(0,0) + 2x^2y\partial^2_{1,2} f(0,0) - xy^2\partial^2_{1,1} f(0,0)\big) \\ & - xy\partial_2 f(0,0) + y^2\partial_1 f(0,0) + \frac{1}{2}\big(-x^2y\partial^2_{1,2} f(0,0) + 2xy^2\partial^2_{1,1} f(0,0) + y^3\partial^2_{1,2} f(0,0)\big) \\ & + o\left(\big(x^2+y^2\big)^{3/2}\right) \\ \underset{(x,y)\to(0,0)}{=} \; & \big(x^2+y^2\big)\partial_1 f(0,0) + \frac{1}{2}\big(x^3\partial^2_{1,1} f(0,0) + \big(x^2y+y^3\big)\partial^2_{1,2} f(0,0) + xy^2\partial^2_{1,1} f(0,0)\big) \\ & + o\left(\big(x^2+y^2\big)^{3/2}\right) \\ \underset{(x,y)\to(0,0)}{=} \; & \big(x^2+y^2\big)\partial_1 f(0,0) + \frac{1}{2}\big(x^2+y^2\big)\big(x\partial^2_{1,1} f(0,0) + y\partial^2_{1,2} f(0,0)\big) + o\left(\big(x^2+y^2\big)^{3/2}\right). \end{aligned}$$

Hence,

$$u(x,y) \underset{(x,y)\to(0,0)}{=} \partial_1 f(0,0) + \frac{1}{2}\big(x\partial^2_{1,1} f(0,0) + y\partial^2_{1,2} f(0,0)\big) + o\left(\big(x^2+y^2\big)^{1/2}\right),$$

as required.

b) Define the mapping

$$\begin{array}{rccc} \alpha : & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & \dfrac{\partial^2_{1,1} f(0,0)}{2} x + \dfrac{\partial^2_{1,2} f(0,0)}{2} y. \end{array}$$

Clearly α is linear, and it follows from the previous question that

$$\frac{u(x,y) - u(0,0) - \alpha(x,y)}{\|(x,y)\|_2} \underset{(x,y)\to(0,0)}{=} o(1) \underset{(x,y)\to(0,0)}{\longrightarrow} 0,$$

hence u is differentiable at $(0,0)$ and $\mathrm{d}_{(0,0)} u = \alpha$.

Exercise 3.

1. Let $x,y) \in \mathbb{R}^2$. Then:

$$\partial_1 f(x,y) = y^3 g'\big(xy^2\big), \quad \partial_2 f(x,y) = g\big(xy^2\big) + 2xy^2 g'\big(xy^2\big),$$
$$\partial^2_{1,1} f(x,y) = y^5 g''\big(xy^2\big), \quad \partial^2_{1,2} f(x,y) = 3y^2 g'\big(xy^2\big) + 2xy^4 g''\big(xy^2\big), \quad \partial^2_{2,2} f(x,y) = 6xyg'\big(xy^2\big) + 4x^2y^3 g''\big(xy^2\big).$$

2. We hence have:

$$f(2,1) = g(2) = 3, \qquad \partial_1 f(2,1) = g'(2) = -1, \qquad \begin{aligned} \partial_2 f(2,1) &= g(2) + 4g'(2) \\ &= 3 - 4 = -1, \end{aligned}$$

$$\partial^2_{1,1} f(2,1) = g''(2) = 1, \qquad \begin{aligned} \partial^2_{1,2} f(2,1) &= 3g'(2) + 4g''(2) \\ &= -3 + 4 = 1, \end{aligned} \qquad \begin{aligned} \partial^2_{2,2} f(2,1) &= 12g'(2) + 16g''(2) \\ &= -12 + 16 = 4. \end{aligned}$$

and hence

$$f(2+h, 1+k) \underset{(h,k)\to(0,0)}{=} 3 - h - k + \frac{1}{2}h^2 + hk + 2k^2 + o\big(h^2 + k^2\big).$$

3. The gradient of f at $(2,1)$ gives a normal vector to $\mathscr{C}$ at $(2,1)$. Now, $\vec{\nabla} f(2,1) = (-1,-1)$, hence an equation of Δ is:

$$(\Delta) \qquad -(x-2) - (y-1) = 0.$$

4. a) Since $f(2,1) = 3$, we must have $\varphi(2) = 1$. Moreover, differentiating $(**)$ yields

$$\forall x \in \mathbb{R},\ \partial_1 f\big(x, \varphi(x)\big) + \varphi'(x)\partial_2 f\big(x, \varphi(x)\big) = 0,$$

hence

$$\partial_1 f\big(2, \varphi(2)\big) + \varphi'(2)\partial_2 f\big(2, \varphi(2)\big) = 0,$$

hence, since $\varphi(2) = 1$, we must have $-1 - \varphi'(2) = 0$, and hence

$$\varphi'(2) = -1.$$

Differentiating again we obtain

$$\forall x \in \mathbb{R},\ \partial^2_{1,1} f\big(x, \varphi(x)\big) + 2\varphi'(x)\partial^2_{1,2} f\big(x, \varphi(x)\big) + \varphi''(x)\partial_2 f\big(x, \varphi(x)\big) + \varphi'(x)^2 \partial^2_{2,2} f\big(x, \varphi(x)\big) = 0.$$

and hence, evaluating at $x = 2$ and using $\varphi(2) = 1$ and $\varphi'(2) = -1$ we obtain: $\varphi''(2) = 3$.

b) Hence the Taylor–Young expansion of φ at 2 is

$$\varphi(2+h) \underset{h\to 0}{=} 1 - h + \frac{3}{2}h^2 + o\big(h^2\big),$$

and we conclude that $\mathscr{C}$ lies above its tangent line in a neighborhood of $(2,1)$. See figure 2.

Exercise 4.

1. a) Let $(x,y) \in (-1,1) \times \mathbb{R}$. Since $x \in (-1,1)$ we have $x^2 < 1$ and hence $x^4 \leq x^2$. Hence

$$x^2 + y^4 \geq x^4 + y^4 = \|(x,y)\|_4^4.$$

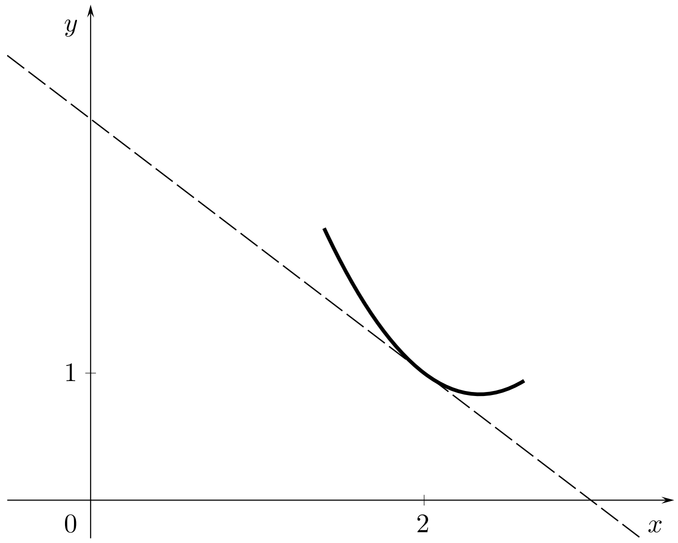

Figure 2. Sketch of the curve $\mathscr{C}$ and Δ of Exercise 3 in a neighborhood of $(2,1)$

b) Hence, for $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$,

$$\left|f(x,y) - f(0,0)\right| = f(x,y) \leq \frac{\left\|(x,y)\right\|_2^{9/2}}{\left\|(x,y)\right\|_4^4}.$$

Now, since $\mathbb{R}^2$ is a finite-dimensional vector space, the 2-norm and the 4-norm are equivalent; in particular, there exists $\alpha > 0$ such that $\|\cdot\|_2 \leq \alpha\|\cdot\|_4$, hence

$$\left|f(x,y) - f(0,0)\right| \leq \frac{\left\|(x,y)\right\|_2^{9/2}}{\left\|(x,y)\right\|_4^4} \leq \frac{\alpha^{9/2}\left\|(x,y)\right\|_4^{9/2}}{\left\|(x,y)\right\|_4^4} = \alpha^{9/2}\left\|(x,y)\right\|_4^{1/2} \underset{(x,y)\to(0,0)}{\longrightarrow} 0,$$

and we conclude that f is continuous at $(0,0)$.

2. Let $h \in \mathbb{R}^*$. Then

$$\frac{f(h,0) - f(0,0)}{h} = \frac{|h|^{9/2}}{h^3} = \frac{|h|^{5/2}}{h} = |h|^{3/2}\frac{|h|}{h} \underset{h\to 0}{\longrightarrow} 0,$$

since $|h|/h$ remains bounded and $|h|^{3/2} \underset{h\to 0}{\longrightarrow} 0$. Hence $\partial_1 f(0,0)$ exists and $\partial_1 f(0,0) = 0$.

Let $h \in \mathbb{R}^*$. Then

$$\frac{f(0,h) - f(0,0)}{h} = \frac{|h|^{9/2}}{h^5} = \frac{|h|^{1/2}}{h},$$

the limit of which doesn't exist as $h \to 0$ (it's easy to see that the right-sided limit yields $+\infty$). Hence $\partial_2 f(0,0)$ doesn't exist.

Since $\partial_2 f(0,0)$ doesn't exist, f is not differentiable at $(0,0)$.

3. Let $(\alpha,\beta) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and let $t \in \mathbb{R}^*$. Then

$$\begin{aligned}\frac{f(t\alpha,t\beta) - f(0,0)}{t} &= \frac{\left(t^2\alpha^2 + t^2\beta^2\right)^{9/4}}{t\left(t^2\alpha^2 + t^4\beta^4\right)} = \frac{|t|^{9/2}\left(\alpha^2+\beta^2\right)^{9/4}}{t^3\left(\alpha^2 + t^2\beta^4\right)} = \frac{|t|^{5/2}\left(\alpha^2+\beta^2\right)^{9/4}}{t\left(\alpha^2 + t^2\beta^4\right)}\\ &= \begin{cases} \dfrac{|t|^{5/2}\left(\alpha^2+\beta^2\right)^{9/4}}{t\left(\alpha^2 + t^2\beta^4\right)} & \text{if } \alpha \neq 0\\ \dfrac{|t|^{1/2}|\beta|^{9/2}}{t\beta^4} & \text{if } \alpha = 0\end{cases}\\ &\underset{t\to 0}{\longrightarrow} \begin{cases} 0 & \text{if } \alpha \neq 0\\ \text{DNE} & \text{if } \alpha = 0.\end{cases}\end{aligned}$$

Hence the directional derivatives of f at $(0,0)$ exist in all direction except in a direction of the form $(0,\beta)$ for $\beta \in \mathbb{R}^*$.