INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — S1 — Math Test #2 — 2h

December 11, 2017

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. Let

$$D = \{(u,v) \in \mathbb{R}^2 \mid u - v > 0\} \qquad \text{and} \qquad \Omega = \mathbb{R}_+^* \times \mathbb{R}.$$

You're given that D and Ω are open subsets of $\mathbb{R}^2$ (but this should be obvious to you). We define the mapping

$$\begin{array}{rccc} \varphi : & D & \longrightarrow & \Omega \\ & (u,v) & \longmapsto & \left(e^{u+v}, \ln(u-v)\right). \end{array}$$

1. a) Check (very briefly) that φ is well-defined.
   b) Show that φ is a C^1-diffeomorphism, and determine φ^{-1} explicitly.
   c) i) For $(u,v) \in D$, express the Jacobian matrix $J_{(u,v)}\varphi$ of φ at (u,v).
      ii) For $(x,y) \in D$, express the Jacobian matrix $J_{(x,y)}(\varphi^{-1})$ of φ^{-1} at (x,y).
      iii) What relation exists between $J\varphi$ and $J(\varphi^{-1})$ (where the Jacobian matrices are taken at appropriate points)? Check explicitly that this is indeed the case.
2. Use φ to find all functions $f : \Omega \to \mathbb{R}$ of class C^1 such that

$$(*) \qquad \forall (x,y) \in \Omega,\ xe^y \partial_1 f(x,y) - \partial_2 f(x,y) - 3e^y f(x,y) = 0.$$

Exercise 2. Let U be a simply-connected set of $\mathbb{R}^2$ such that $(0,0) \in U$. Let $f : U \to \mathbb{R}$ be function of class C^2, and assume moreover that f satisfies the following equation:[2]

$$\text{(H)} \qquad \partial^2_{1,1} f + \partial^2_{2,2} f = 0.$$

We define the differential form ω on U by

$$\forall (x,y) \in U,\ \omega_{(x,y)} = -\partial_2 f(x,y) e_1' + \partial_1 f(x,y) e_2',$$

where (e_1', e_2') is the dual basis of $\mathbb{R}^2$.

1. Show that there exists a function $g : U \to \mathbb{R}$ of class C^2 such that $dg = \omega$ and $g(0,0) = 0$.
2. Show that g is harmonic, i.e., that $\partial^2_{1,1} g + \partial^2_{2,2} g = 0$. Determine the second order partial derivative $\partial^2_{1,2} g$ of g in terms of the partial derivatives of f only.
3. We define the function u as

$$\begin{array}{rccc} u : & U & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & \begin{cases} \dfrac{xf(x,y) + yg(x,y)}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ \partial_1 f(0,0) & \text{if } (x,y) = (0,0). \end{cases} \end{array}$$

   a) We now assume that $f(0,0) = 0$. Use the second order Taylor–Young expansion of f and g at $(0,0)$ to show that[3]

$$u(x,y) \underset{(x,y)\to(0,0)}{=} \partial_1 f(0,0) + \frac{1}{2} x \partial^2_{1,1} f(0,0) + \frac{1}{2} y \partial^2_{1,2} f(0,0) + o\left(\sqrt{x^2+y^2}\right).$$

   b) Deduce that u is differentiable at $(0,0)$ and explicit the differential $d_{(0,0)}u$ of u at $(0,0)$.

[2] such a function f is called a *harmonic function*.

[3] You may use, without any justifications, that

$$x o(x^2+y^2) \underset{(x,y)\to(0,0)}{=} o\left((x^2+y^2)^{3/2}\right) \quad \text{and} \quad y o(x^2+y^2) \underset{(x,y)\to(0,0)}{=} o\left((x^2+y^2)^{3/2}\right).$$

Exercise 3. Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a function of class C^2 such that

$$g(2) = 3, \qquad g'(2) = -1, \qquad g''(2) = 1,$$

and define the function f as[4]

$$f : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x, y) & \longmapsto & yg(xy^2). \end{array}$$

1. Determine the following first and second order partial derivatives of f

$$\partial_1 f, \quad \partial_2 f, \quad \partial^2_{1,1} f, \quad \partial^2_{1,2} f, \quad \partial^2_{2,2} f$$

   in terms of g and its derivatives at well-chosen points.

2. Deduce the second order Taylor–Young expansion of f at $(2, 1)$.

3. We denote by $\mathscr{C}$ the level set of f at level 3, i.e.,

$$\mathscr{C} = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 3\}.$$

   We assume that $\mathscr{C}$ is a curve. Give an equation of the tangent line Δ to $\mathscr{C}$ at $(2, 1)$.

4. We assume, moreover, that there exists $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ of class C^2 such that $\mathscr{C}$ possesses a representation of the form

$$(\mathscr{C}) \qquad y = \varphi(x),$$

   that is,

$$(*) \qquad \forall (x, y) \in \mathbb{R}^2, \ f(x, y) = 3 \iff y = \varphi(x).$$

   Note that, in particular, we must have

$$(**) \qquad \forall x \in \mathbb{R}, \ f(x, \varphi(x)) = 3.$$

   a) Determine the value of $\varphi(2)$, $\varphi'(2)$ and $\varphi''(2)$. *Hint: to determine the value of $\varphi'(2)$ and $\varphi''(2)$ you might want to differentiate $(**)$.*

   b) Deduce the relative position of $\mathscr{C}$ with respect to Δ in a neighborhood of $(2, 1)$, and sketch the curve $\mathscr{C}$ in a neighborhood of $(2, 1)$.

Exercise 4. Let

$$f : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x, y) & \longmapsto & \begin{cases} \dfrac{(x^2 + y^2)^{9/4}}{x^2 + y^4} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases} \end{array}$$

1. In this question we show that f is continuous at $(0, 0)$.

   a) Briefly check that[5]

$$\forall (x, y) \in (-1, 1) \times \mathbb{R}, \ x^2 + y^4 \geq \|(x, y)\|_4^4.$$

   b) Deduce that f is continuous at $(0, 0)$.

2. Study the existence of the partial derivatives $\partial_1 f(0, 0)$ and $\partial_2 f(0, 0)$, and determine their value if they exist. Is f differentiable at $(0, 0)$?

3. Let $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Determine whether the directional derivative $\nabla_{(\alpha,\beta)} f(0, 0)$ of f at $(0, 0)$ in the direction (α, β) exists, and determine its value if it exists.

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[4] The function f thus defined is obviously of class C^2, so you don't have to justify this fact.

[5] We recall that for $p \in [1, +\infty)$, the p-norm $\|\cdot\|_p$ defined on $\mathbb{R}^2$ by

$$\forall (x, y) \in \mathbb{R}^2, \ \|(x, y)\|_p = (|x|^p + |y|^p)^{1/p}$$

is a norm on $\mathbb{R}^2$.