INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Solution of Math Test #3

Romaric Pujol, romaric.pujol@insa-lyon.fr

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Exercise 1.

1. We use the Implicit Function Theorem:
   - The point $(1,0)$ belongs to $\mathscr{C}$ since $f(1,0)=1$.
   - The function f is of class C^∞.
   - $\partial_2 f(0,0)=-3\neq 0$.

   Hence, by the Implicit Function Theorem, there exists a neighborhood U of 1 in $\mathbb{R}$ and a neighborhood V of 0 in $\mathbb{R}$, and a function $\varphi : U\to V$ of class C^∞ such that

   $$\forall (x,y)\in U\times V,\ \big(f(x,y)=0 \iff y=\varphi(x)\big).$$

   Hence, the intersection of $\mathscr{C}$ with $U\times V$ is the graph of φ.

2. a) i) $a=0$.

   ii) We know that $\varphi(1)=0$ and that

   $$\forall x\in U,\ \varphi'(x)=-\frac{\partial_1 f\big(x,\varphi(x)\big)}{\partial_2 f\big(x,\varphi(x)\big)}.$$

   Now, for $x,y\in\mathbb{R}$,

   $$\partial_1 f(x,y)=3x^2-3y,\qquad \partial_2 f(x,y)=3y^2-3x.$$

   Hence,

   $$\forall x\in U,\ \varphi'(x)=-\frac{x^2-\varphi(x)}{\varphi(x)^2-x}.$$

   Hence,

   $$\varphi'(1)=1.$$

   We now differentiate φ': for $x\in U$,

   $$\varphi''(x)=-\frac{\big(2x-\varphi'(x)\big)\big(\varphi(x)^2-x\big)-\big(x^2-\varphi(x)\big)\big(2\varphi'(x)\varphi(x)-1\big)}{\big(\varphi(x)^2-x\big)^2}.$$

   Evaluating at 1, using $\varphi(1)=0$ and $\varphi'(1)=1$ yields

   $$\varphi''(1)=0.$$

   Hence $b=\varphi'(1)=1$ and $c=\varphi''(1)/2=0$.

   b)

   $$\begin{aligned}1=f\big(1+h,\varphi(1+h)\big)&\underset{h\to0}{=}(1+h)^3+\Big(h+dh^3+o\big(h^3\big)\Big)^3-3(1+h)\Big(h+dh^3+o\big(h^3\big)\Big)\\&\underset{h\to0}{=}1+3h+3h^2+h^3+h^3-3h-3dh^3-3h^2+o\big(h^3\big)\\&\underset{h\to0}{=}1+(2-3d)h^3+o\big(h^3\big)\end{aligned}$$

   Hence $d=2/3$.

   c) Hence the Taylor–Young expansion of φ at 1 is

   $$\varphi(x)\underset{x\to1}{=}(x-1)+\frac{2}{3}(x-1)^3+o\big((x-1)^3\big),$$

   and we conclude that an equation of the tangent line to φ at $A(1,0)$ is is

   $$\Delta\colon y=x-1$$

   and the graph of φ (and hence $\mathscr{C}$) lies:
   - Above Δ for $x>1$ (and x close enough to 1),

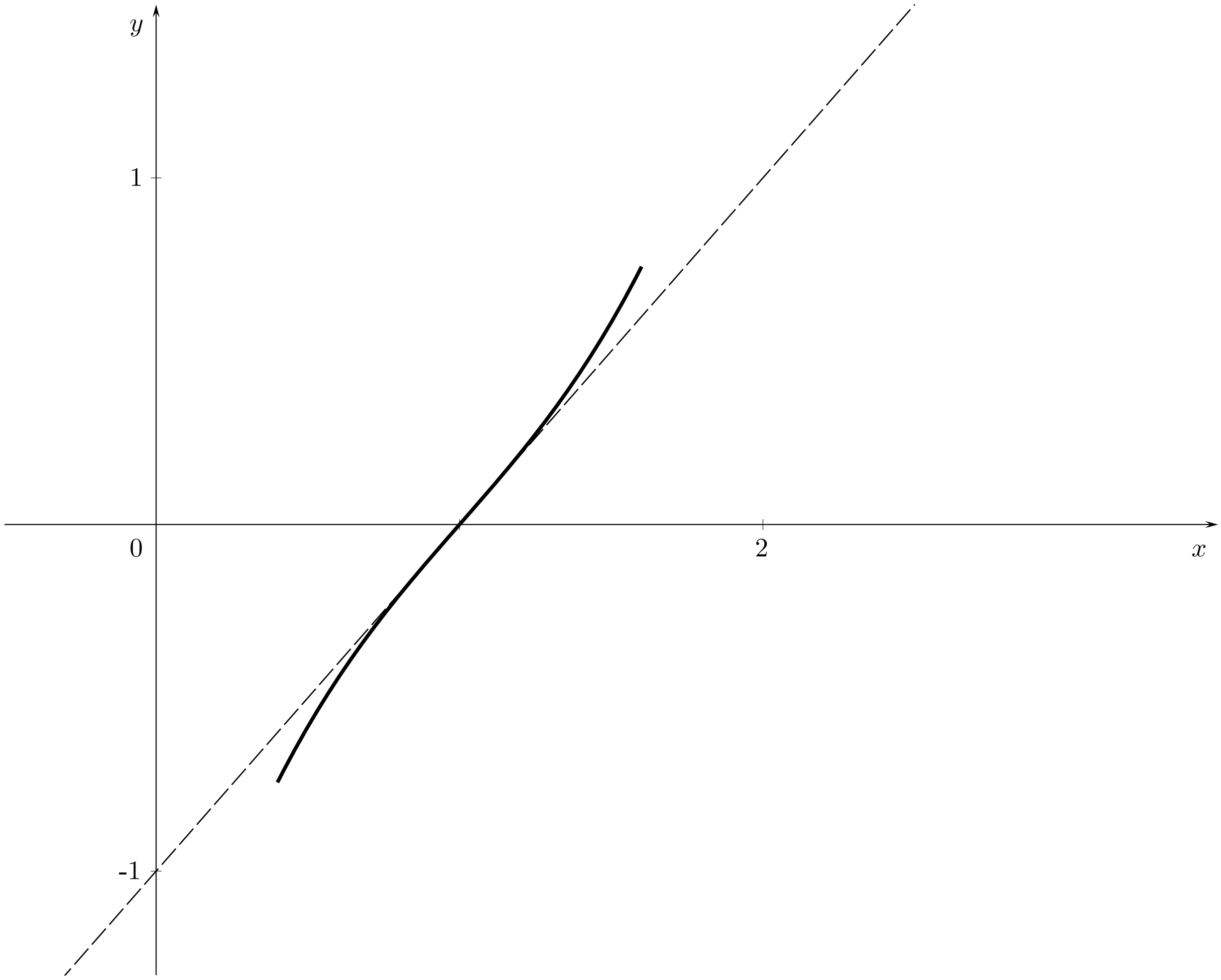

Figure 3. Curve $\mathscr{C}$ is a neighborhood of $A(1,0)$, Exercise 1. The curve $\mathscr{C}$ crosses its tangent line at $A(1,0)$, hence A is a point of inflection of $\mathscr{C}$.

- Below Δ for $x<1$ (and x close enough to 1),

hence $A(1,0)$ is indeed a point of inflection of $\mathscr{C}$.

d) See Figure 3.

Exercise 2.

1. We use the Implicit Function Theorem:

   - The point M_0 belongs to $\mathscr{S}$ since $f(1,0,0)=1$.
   - The function f is of class C^∞.
   - $\partial_2 f(1,0,0)=1\neq 0$.

   Hence, by the Implicit Function Theorem, there exists a neighborhood U of $(1,0)$ in $\mathbb{R}^2$ and a neighborhood V of 0 in $\mathbb{R}$ and a function $\varphi : U\to V$ such that

   $$\forall (x,z)\in U,\ \forall y\in V,\ \big(f(x,y,z)=1 \iff y=\varphi(x,z)\big).$$

2. Moreover, we know that $\varphi(1,0)=0$ and for all $(x,z)\in U$,

   $$\partial_1\varphi(x,z)=-\frac{\partial_1 f\big(x,\varphi(x,z),z\big)}{\partial_2 f\big(x,\varphi(x,z),z\big)}$$
   $$\partial_2\varphi(x,z)=-\frac{\partial_3 f\big(x,\varphi(x,z),z\big)}{\partial_2 f\big(x,\varphi(x,z),z\big)}.$$

Now, for $(x,y,z)\in\mathbb{R}^3$,

$$\begin{aligned}\partial_1 f(x,y,z) &= \big(1+(x+y+z)yz\big)\mathrm{e}^{xyz},\\ \partial_2 f(x,y,z) &= \big(1+(x+y+z)xz\big)\mathrm{e}^{xyz},\\ \partial_3 f(x,y,z) &= \big(1+(x+y+z)xy\big)\mathrm{e}^{xyz},\end{aligned}$$

hence, for all $(x,z)\in U$,

$$\begin{aligned}\partial_1\varphi(x,z) &= -\frac{1+\big(x+\varphi(x,z)+z\big)\varphi(x,z)z}{1+\big(x+\varphi(x,z)+z\big)xz},\\ \partial_2\varphi(x,z) &= -\frac{1+\big(x+\varphi(x,z)+z\big)x\varphi(x,z)}{1+\big(x+\varphi(x,z)+z\big)xz}.\end{aligned}$$

Hence

$$\partial_1\varphi(1,0)=-1,\qquad\qquad \partial_2\varphi(1,0)=-1.$$

We now compute the second order partial derivatives of φ at $(1,0)$, using the rate of change:

- $\partial^2_{1,1}\varphi(1,0)$: for $h\neq 0$ such that $(1+h,0)\in U$:

  $$\frac{\partial_1\varphi(1+h,0)-\partial_1\varphi(1,0)}{h}=0\underset{h\to 0}{\longrightarrow}0.$$

  Hence $\partial^2_{1,1}\varphi(1,0)=0$.
- $\partial^2_{1,2}\varphi(1,0)$: for $h\neq 0$ such that $(1+h,0)\in U$:

  $$\begin{aligned}\frac{\partial_2\varphi(1+h,0)-\partial_2\varphi(1,0)}{h} &= -\frac{\big(1+h+\varphi(1+h,0)\big)(1+h)\varphi(1+h,0)}{h}\\ &= -\big(1+h+\varphi(1+h,0)\big)(1+h)\frac{\varphi(1+h,0)}{h}\\ &\underset{h\to 0}{\longrightarrow} -\partial_1\varphi(1,0)=1.\end{aligned}$$

  Hence $\partial^2_{1,2}\varphi(1,0)=-1$.
- $\partial^2_{2,2}\varphi(1,0)$: for $h\neq 0$ such that $(1,h)\in U$:

  $$\begin{aligned}\frac{\partial_2\varphi(1,h)-\partial_2\varphi(1,0)}{h} &= \frac{1}{h}\left(-\frac{1+\big(1+\varphi(1,h)+h\big)\varphi(1,h)}{1+\big(1+\varphi(1,h)+h\big)h}+1\right)\\ &= \frac{1}{h}\left(-\frac{\big(1+\varphi(1,h)+h\big)\big(\varphi(1,h)-h\big)}{1+\big(1+\varphi(1,h)+h\big)h}\right)\\ &= -\frac{\big(1+\varphi(1,h)+h\big)}{1+\big(1+\varphi(1,h)+h\big)h}\,\frac{\big(\varphi(1,h)-h\big)}{h}\\ &\underset{h\to 0}{\longrightarrow} -\big(\partial_2\varphi(1,0)-1\big)=2.\end{aligned}$$

  Hence $\partial^2_{2,2}\varphi(1,0)=2$.

We hence conclude that the second order Taylor–Young expansion of φ at $(1,0)$ is:

$$\varphi(1+h,k)\underset{(h,k)\to(0,0)}{=}-h-k+hk+k^2+o\big(h^2+k^2\big).$$

Exercise 3.

1.
   - For $(u,v)\in\Omega$, u/v is well-defined and positive, hence $\ln(u/v)$ is well-defined. Moreover, $v>0$, hence $\big(v,\ln(u/v)\big)\in\mathbb{R}_+^*\times\mathbb{R}=D$. Hence φ is well-defined.
   - φ is clearly of class C^2 (in fact even of class C^∞).

- We now show that φ is a bijection: let $(u,v)\in\Omega$ and $(x,y)\in D$. Then

$$\varphi(u,v)=(x,y)\iff\begin{cases}v=x\\ \ln(u/v)=y\end{cases}\iff\begin{cases}v=x\\ u/v=\mathrm{e}^y\end{cases}\iff\begin{cases}v=x\\ u=x\mathrm{e}^y.\end{cases}$$

Now for $(x,y)\in D=\mathbb{R}_+^*\times\mathbb{R}$, $\bigl(x\mathrm{e}^y,x\bigr)\in\mathbb{R}_+^*\times\mathbb{R}_+^*=\Omega$, hence φ is a bijection, and

$$\varphi^{-1}\ :\ \begin{array}{ccc}D&\longrightarrow&\Omega\\ (x,y)&\longmapsto&\bigl(x\mathrm{e}^y,x\bigr).\end{array}$$

- Clearly, φ^{-1} is of class C^2 (and even of class C^∞).

Hence φ is a C^2-diffeomorphism from Ω to D. Hence we can use φ to solve $(*)$.

2. Let $f:D\to\mathbb{R}$ be a function, and let $g=f\circ\varphi$. Since φ is a C^2-diffeomorphism,

$$f\text{ is of class }C^2\iff g\text{ is of class }C^2.$$

More explicitly, the relation between f and g is:

$$\forall(x,y)\in D,\ f(x,y)=g\bigl(x\mathrm{e}^y,x\bigr).$$

We now assume that f is of class C^2 (so that g is also of class C^2; we'll use Schwarz' Theorem without explicitly mentioning it). Let $(x,y)\in D$ and set $(u,v)=\varphi^{-1}(x,y)=\bigl(x\mathrm{e}^y,x\bigr)$. We express the partial derivatives of f in terms of that of g:

$$\begin{aligned}\partial_1 f(x,y)&=\mathrm{e}^y\partial_1 g\bigl(x\mathrm{e}^y,x\bigr)+\partial_2 g\bigl(x\mathrm{e}^y,x\bigr),\\ \partial_2 f(x,y)&=x\mathrm{e}^y\partial_1 g\bigl(x\mathrm{e}^y,x\bigr),\\ \partial_{1,1}^2 f(x,y)&=\mathrm{e}^{2y}\partial_{1,1}^2 g(u,v)+\mathrm{e}^y\partial_{2,1}^2 g(u,v)+\mathrm{e}^y\partial_{1,2}^2 g(u,v)+\partial_{2,2}^2 g(u,v)\\ &=\mathrm{e}^{2y}\partial_{1,1}^2 g(u,v)+2\mathrm{e}^y\partial_{1,2}^2 g(u,v)+\partial_{2,2}^2 g(u,v),\\ \partial_{1,2}^2 f(x,y)&=\mathrm{e}^y\partial_1 g(u,v)+x\mathrm{e}^{2y}\partial_{1,1}^2 g(u,v)+x\mathrm{e}^y\partial_{2,1}^2 g(u,v),\\ \partial_{2,2}^2 f(x,y)&=x\mathrm{e}^y\partial_1 g(u,v)+x^2\mathrm{e}^{2y}\partial_{1,1}^2 g(u,v).\end{aligned}$$

Hence,

$$\begin{aligned}&x^2\partial_{1,1}^2 f(x,y)-2x\partial_{1,2}^2 f(x,y)+\partial_{2,2}^2 f(x,y)+\partial_2 f(x,y)+x^2 f(x,y)\\ &\qquad=x^2\mathrm{e}^{2y}\partial_{1,1}^2 g(u,v)+2x^2\mathrm{e}^y\partial_{1,2}^2 g(u,v)+x^2\partial_{2,2}^2 g(u,v)\\ &\qquad\quad-2x\mathrm{e}^y\partial_1 g(u,v)-2x^2\mathrm{e}^{2y}\partial_{1,1}^2 g(u,v)-2x^2\mathrm{e}^y\partial_{2,1}^2 g(u,v)\\ &\qquad\quad+x\mathrm{e}^y\partial_1 g(u,v)+x^2\mathrm{e}^{2y}\partial_{1,1}^2 g(u,v)\\ &\qquad\quad+x\mathrm{e}^y\partial_1 g(u,v)\\ &\qquad\quad+x^2 g(u,v)\\ &\qquad=v^2\partial_{2,2}^2 g(u,v)+v^2 g(u,v).\end{aligned}$$

Hence,

$$\begin{aligned}f\text{ is a solution of }(*)\text{ on }D&\iff\forall(u,v)\in\Omega,\ v^2\partial_{2,2}^2 g(u,v)+v^2 g(u,v)=v^2\\ &\iff\forall(u,v)\in\Omega,\ \partial_{2,2}^2 g(u,v)+g(u,v)=1.\end{aligned}$$

Let $u_0\in\mathbb{R}_+^*$ and define the function h as

$$h\ :\ \begin{array}{ccc}\mathbb{R}_+^*&\longrightarrow&\mathbb{R}\\ v&\longmapsto&g(u_0,v).\end{array}$$

Clearly,

$$\partial_{2,2}^2 g(u_0,v)+g(u_0,v)=h''(v)+h(v).$$

Now we know that the general solution of the differential equation $h''+h=1$ is

$$h(v)=A\cos(v)+B\sin(v)+1,$$

for $A, B \in \mathbb{R}$.

Hence,

$$\begin{aligned} f \text{ is a solution of } (*) \text{ on } D &\iff \exists A\colon \mathbb{R}_+^* \to \mathbb{R},\ \exists B\colon \mathbb{R}_+^* \to \mathbb{R}, \\ &\qquad \forall (u,v) \in \Omega,\ g(u,v) = A(u)\cos(v) + B(u)\sin(v) + 1 \\ &\iff \exists A\colon \mathbb{R}_+^* \to \mathbb{R},\ \exists B\colon \mathbb{R}_+^* \to \mathbb{R}, \\ &\qquad \forall (x,y) \in D,\ f(x,y) = A\big(x\mathrm{e}^y\big)\cos(x) + B\big(x\mathrm{e}^y\big)\sin(x) + 1 \end{aligned}$$

Hence, the general solution of class C^2 of $(*)$ on D is of the form

$$f(x,y) = A\big(x\mathrm{e}^y\big)\cos(x) + B\big(x\mathrm{e}^y\big)\sin(x) + 1$$

for $A\colon \mathbb{R}_+^* \to \mathbb{R}$ and $B\colon \mathbb{R}_+^* \to \mathbb{R}$ of class C^2.

Exercise 4.

1. (1) Series (1) is a geometric series of ratio $\mathrm{e}^{-1} \in (-1, 1)$, hence is convergent.
   (2) Let $n \in \mathbb{N}$. We have $n^2 \geq n$, hence $0 \leq \mathrm{e}^{-n^2} \leq \mathrm{e}^{-n}$. Hence, by the comparison test (and since Series (1) is convergent), Series (2) is convergent.
   (3) We have $\lim_{n\to+\infty} \mathrm{e}^{-1/n^2} = 1 \neq 0$, hence Series (3) is divergent.

2. Since $\alpha > 0$, $\dfrac{1}{n^{2\alpha}} \underset{n\to+\infty}{\longrightarrow} 0$ and we hence have the following equivalent:

$$\exp\left(\frac{1}{n^{2\alpha}}\right) - 1 \underset{n\to+\infty}{\sim} \frac{1}{n^{2\alpha}} > 0.$$

   Now, the sequence $\left(\dfrac{1}{n^{2\alpha}}\right)_{n\geq 1}$ is the general term of a convergent series if and only if $2\alpha > 1$ if and only if $\alpha > 1/2$ (Riemann). Hence, by the equivalent test, the series is convergent if and only if $\alpha > 1/2$.

3. For $n \in \mathbb{N}$ set

$$u_n = \frac{n}{2^n + 1}.$$

   Clearly the sequence $(u_n)_{n\in\mathbb{N}}$ is a sequence with positive terms. Now, for $n \in \mathbb{N}$,

$$\frac{u_{n+1}}{u_n} = \frac{n+1}{2^{n+1}+1}\,\frac{2^n+1}{n} \underset{n\to+\infty}{\sim} \frac{n}{2^{n+1}}\,\frac{2^n}{n} = \frac{1}{2} \underset{n\to+\infty}{\longrightarrow} \frac{1}{2} < 1.$$

   Hence, by the ratio test, the series converges.

4. a) For $n \in \mathbb{N}$ define

$$u_n = \frac{(-1)^n}{2n+1}.$$

   - Clearly, the sequence $(u_n)_{n\in\mathbb{N}}$ is an alternating sequence.
   - For $n \in \mathbb{N}$, $|u_{n+1}| = \dfrac{1}{2n+3} \leq \dfrac{1}{2n+1} = |u_n|$.
   - $\lim_{n\to+\infty} u_n = 0$.

   Hence, by the alternating series test, the series $\sum_n u_n$ is convergent.

   b) Let $N \in \mathbb{N}^*$. By the alternating series test, we moreover know that

$$|S - S_N| = \left|\sum_{n=N+1}^{+\infty} u_n\right| \leq |u_{N+1}|.$$

   Hence it's sufficient to choose N such that $|u_{N+1}| \leq 10^{-3}$. Now,

$$|u_{N+1}| \leq 10^{-3} \iff \frac{1}{2N+3} \leq 10^{-3} \iff 2N + 3 \geq 1000 \iff 2N \geq 997,$$

   so we can choose $N = 499$.

5. We denote by $(u_n)_{n\in\mathbb{N}^*}$ the general term of the series. Since $\alpha > 0$, we have $\lim_{n\to+\infty} \frac{(-1)^n}{n^\alpha} = 0$, hence

$$u_n = \exp\left(\frac{(-1)^n}{n^\alpha}\right) - 1 \underset{n\to+\infty}{=} \frac{(-1)^n}{n^\alpha} + \frac{1}{2n^{2\alpha}} + o\left(\frac{1}{n^{2\alpha}}\right).$$

For $n \in \mathbb{N}^*$, define

$$a_n = \frac{(-1)^n}{n^\alpha}, \qquad b_n = u_n - a_n.$$

Clearly, the sequence $(a_n)_{n\in\mathbb{N}^*}$ is the general term of a convergent alternating Riemann series (since $\alpha > 0$). Now,

$$b_n \underset{n\to+\infty}{\sim} \frac{1}{2n^{2\alpha}} > 0,$$

and $(b_n)_{n\in\mathbb{N}^*}$ is the general term of a convergent series if and only if $2\alpha > 1$, if and only if $\alpha > 1/2$. Hence, by the equivalent test, the series $\sum_n b_n$ converges if and only if $\alpha > 1/2$. We hence conclude that the series $\sum_n u_n$ converges if and only if $\alpha > 1/2$.