SCAN 2 — S1 — Math Test #3 — 2h

January 22, 2018

No documents, no calculators, no cell phones, no electronic devices, no free variables allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. Let

$$f: \begin{array}{rcl} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x, y) & \longmapsto & x^3 + y^3 - 3xy, \end{array}$$

and let $\mathscr{C}$ be the curve of $\mathbb{R}^2$ defined by the following implicit equation:

$$\mathscr{C}: f(x, y) = 1.$$

1. Show that in the neighborhood of $A(1, 0)$, the curve $\mathscr{C}$ is the graph of a function φ of class C^∞.

2. In this question, we determine the third order Taylor–Young expansion of φ at 1. Since φ is three times differentiable at 1, there exists $a, b, c, d \in \mathbb{R}$ such that:

$$\varphi(1 + h) \underset{h \to 0}{=} a + bh + ch^2 + dh^3 + o(h^3).$$

   a) i) Give the value of a.

      ii) Use the moreover part of the Implicit Function Theorem to determine the value of $\varphi'(1)$ and $\varphi''(1)$ and deduce the value of b and c.

   b) Use the fact that

$$\forall x \in U, \ f\big(x, \varphi(x)\big) = 1$$

   to determine the value of d. More precisely: you're asked to compute $f\big(1 + h, \varphi(1 + h)\big)$ up to $o(h^3)$ as $h \longrightarrow 0$, and identify this expansion with 1 to determine the value of d.

   c) Deduce that $\mathscr{C}$ has a point of inflection at $A(1, 0)$. We recall that a point of inflection of $\mathscr{C}$ is a point where $\mathscr{C}$ crosses its tangent line.

   d) Sketch the curve $\mathscr{C}$ in a neighborhood of $A(1, 0)$.

Exercise 2. Let

$$f: \begin{array}{rcl} \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ (x, y, z) & \longmapsto & (x + y + z)e^{xyz}, \end{array}$$

and let $\mathscr{S}$ be the surface of implicit equation

$$\mathscr{S}: f(x, y, z) = 1.$$

1. Show that, in the neighborhood of $M_0(1, 0, 0)$, the surface $\mathscr{S}$ has an equation of the form $y = \varphi(x, z)$, where φ is of class C^∞.

2. Determine the second order Taylor–Young expansion of φ at $(1, 0)$.

Exercise 3. Let $\Omega = \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $D = \mathbb{R}_+^* \times \mathbb{R}$.

$$\begin{array}{rccc} \varphi : & \Omega & \longrightarrow & D \\ & (u,v) & \longmapsto & \big(v, \ln(u/v)\big). \end{array}$$

The goal of this exercise it find all the functions f of class C^2 on D that are solution of the following partial differential equation, using φ.

$$(*) \qquad x^2\partial_{1,1}^2 f(x,y) - 2x\partial_{1,2}^2 f(x,y) + \partial_{2,2}^2 f(x,y) + \partial_2 f(x,y) + x^2 f(x,y) = x^2.$$

1. Show that φ is a suitable change of coordinates to find all functions f of class C^2 on D satisfying Equation $(*)$. More precisely, check that φ is well-defined and show that φ is a diffeomorphism of appropriate class.
2. Determine all the functions f of class C^2 on D that are solution of Equation $(*)$, using φ.

Exercise 4. The questions of this exercise are independent from each other.

1. Are the following series convergent or divergent? (justify your answer).

$$(1)\ \sum_n e^{-n}, \qquad (2)\ \sum_n e^{-n^2}, \qquad (3)\ \sum_n e^{-1/n^2}.$$

2. Determine the values of $\alpha \in \mathbb{R}_+^*$ for which the following series is convergent:

$$\sum_n \left(\exp\left(\frac{1}{n^{2\alpha}}\right) - 1\right).$$

3. Determine, using the ratio test, whether the following series converges or diverges:

$$\sum_n \frac{n}{2^n + 1}.$$

4. a) Check that the alternating series test can be used to show that the following series is convergent:

$$\sum_n \frac{(-1)^n}{2n+1},$$

b) We define

$$S = \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1}, \qquad \text{and } \forall N \in \mathbb{N}^*,\ S_N = \sum_{n=0}^{N} \frac{(-1)^n}{2n+1}.$$

Find $N \in \mathbb{N}$ such that $|S - S_N| < 10^{-3}$.

5. Use an appropriate Taylor expansion to determine the values of $\alpha \in \mathbb{R}_+^*$ for which the following series is convergent.

$$\sum_n \left(\exp\left(\frac{(-1)^n}{n^{\alpha}}\right) - 1\right).$$