INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Solution of Math Test #4

Romaric Pujol, romaric.pujol@insa-lyon.fr

March 26, 2018

Exercise 1.

1. Let $\alpha > 0$. Since $\dfrac{(-1)^n}{n^\alpha} \underset{n\to+\infty}{\longrightarrow} 0$,

$$\exp\left(\frac{(-1)^n}{n^\alpha}\right) - \cos\left(\frac{(-1)^n}{n^\alpha}\right) \underset{n\to+\infty}{=} \left(1 + \frac{(-1)^n}{n^\alpha} + \frac{1}{2n^{2\alpha}} + o\left(\frac{1}{n^{2\alpha}}\right)\right) - \left(1 - \frac{1}{2n^{2\alpha}} + o\left(\frac{1}{n^{2\alpha}}\right)\right)$$
$$\underset{n\to+\infty}{=} \frac{(-1)^n}{n^\alpha} + \frac{1}{n^{2\alpha}} + o\left(\frac{1}{n^{2\alpha}}\right).$$

Now define the sequence $(u_n)_{n\in\mathbb{N}^*}$ and $(v_n)_{n\in\mathbb{N}^*}$ as

$$\forall n \in \mathbb{N}^*, \qquad u_n = \frac{(-1)^n}{n^\alpha}, \qquad v_n = \exp\left(\frac{(-1)^n}{n^\alpha}\right) - \cos\left(\frac{(-1)^n}{n^\alpha}\right) - u_n.$$

The series $\sum_n u_n$ is a convergent series as it's an alternating Riemann series (with $\alpha > 0$). Moreover,

$$v_n \underset{n\to+\infty}{=} \frac{1}{n^{2\alpha}} + o\left(\frac{1}{n^{2\alpha}}\right) \underset{n\to+\infty}{\sim} \frac{1}{n^{2\alpha}} > 0$$

and by the equivalent test, $\sum_n v_n$ is convergent if and only if $2\alpha > 1$ if and only if $\alpha > 1/2$.
Hence, by sum, Series (1) is convergent if and only $\alpha > 1/2$.

2. We use the ratio test (which is valid since Series (2) is a series with positive terms): for $n \in \mathbb{N}$ define $u_n = \dfrac{n!^2}{(2n)!}$. Then:

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)!^2}{(2n+2)!}\frac{(2n)!}{n!^2} = \frac{(n+1)^2}{(2n+1)(2n+2)} \underset{n\to+\infty}{\longrightarrow} \frac{1}{4} < 1.$$

Hence, by the ratio test Series (2) is convergent.

3. We know that

$$\forall x \in (-1,1],\ \ln(1+x) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} x^n.$$

Now,

$$S = -\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n}\left(-\frac{1}{3}\right)^n,$$

hence

$$S = -\ln\left(1 - \frac{1}{3}\right) = -\ln\left(\frac{2}{3}\right) = \ln\left(\frac{3}{2}\right).$$

Exercise 2.

1. For $n \in \mathbb{N}$ define $u_n = \dfrac{(-1)^n}{1+\sqrt{n}}$.

- The sequence $(u_n)_{n\in\mathbb{N}}$ is clearly an alternating series,
- clearly, $u_n \underset{n\to+\infty}{\longrightarrow} 0$,
- clearly, the sequence $\left(|u_n|\right)_{n\in\mathbb{N}}$ is decreasing,

hence, by the Alternating Series Test, the series $\sum_n u_n$ is convergent.

Now $|u_n| \underset{n\to+\infty}{\sim} \dfrac{1}{n^{1/2}}$ hence, by the equivalent test (and Riemann with $\alpha = 1/2 < 1$) we conclude that the series $\sum_n u_n$ is not absolutely convergent.

2. Since $\sum_n u_n$ is an alternating series, we have, for $N \in \mathbb{N}$,

$$\begin{cases} R_N \geq 0 & \text{if } N \text{ is even} \\ R_N \leq 0 & \text{if } N \text{ is odd} \end{cases}$$

(the sign of the remainder is that of its first term), and

$$|R_N| \leq |u_{N+1}| = \frac{1}{1+\sqrt{N+1}}$$

(the absolute value of the remainder is non-greater than that of its first term).

3. From the previous inequality,

$$\text{error}_N = |S - S_N| = |R_N| \leq \frac{1}{1+\sqrt{N+1}},$$

hence it is *sufficient* to find N such that

$$\frac{1}{1+\sqrt{N+1}} \leq 10^{-3}.$$

Now,

$$\frac{1}{1+\sqrt{N+1}} \leq 10^{-3} \iff 1+\sqrt{N+1} \geq 1000 \iff \sqrt{N+1} \geq 999 \iff N+1 \geq 999^2$$
$$\iff N \geq 999^2 - 1 = (999-1)(999+1) = 998 \times 1000 = 998000.$$

Exercise 3.

$$\frac{n^2}{1+n^4} \underset{n\to+\infty}{\sim} \frac{1}{n^2} > 0$$

hence, by the equivalent test (and Riemann with $\alpha = 2 > 1$) we conclude that the series $\sum_n \dfrac{n^2}{1+n^4}$ is convergent.
We use the integral comparison test: define the function

$$f : \begin{array}{ccc} [1,+\infty) & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{x^2}{1+x^4}. \end{array}$$

We show that the function f is non-increasing: for $x \in [1,+\infty)$,

$$f'(x) = \frac{2x(1+x^4) - 4x^5}{\left(1+x^4\right)^2} = 2x\frac{1-x^4}{\left(1+x^4\right)^2} \leq 0.$$

Hence, by the integral comparison test, and for $N \in \mathbb{N}^*$,

$$S - S_N = \sum_{n=N+1}^{+\infty} \frac{n^2}{1+n^4} = \sum_{n=N+1}^{+\infty} f(n) \leq \int_N^{+\infty} f(t)\,\mathrm{d}t.$$

(Notice, moreover that $S - S_N \geq 0$ since S is a series with positive terms). Now, for $N \in \mathbb{N}^*$ and $X > N$,

$$\int_N^X f(t)\,\mathrm{d}t = \int_N^X \frac{t^2}{1+t^4}\,\mathrm{d}t \leq \int_N^X \frac{\mathrm{d}t}{t^2} = \left[-\frac{1}{t^2}\right]_{t=N}^{t=X} = \frac{1}{N} - \frac{1}{X} \underset{X\to+\infty}{\longrightarrow} \frac{1}{N}.$$

Hence, in order to have $|S - S_N| < \epsilon$ it is sufficient to have $N > 1/\epsilon$.

Exercise 4.

1. Let $x \in (-R,R)$. Then,

$$f'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1} = \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n$$

$$f''(x) = \sum_{n=2}^{+\infty} n(n-1)a_n x^{n-2} = \sum_{n=1}^{+\infty}(n+1)na_{n+1}x^{n-1}.$$

Hence,

$$\begin{aligned} xf''(x) + (1+x)f'(x) - \lambda f(x) &= \sum_{n=1}^{+\infty}(n+1)na_{n+1}x^n + \sum_{n=0}^{+\infty}(n+1)a_{n+1}x^n + \sum_{n=1}^{+\infty} na_n x^n - \lambda\sum_{n=0}^{+\infty} a_n x^n \\ &= \sum_{n=0}^{+\infty}(n+1)na_{n+1}x^n + \sum_{n=0}^{+\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{+\infty} na_n x^n - \lambda\sum_{n=0}^{+\infty} a_n x^n \\ &= \sum_{n=0}^{+\infty}\Big((n+1)na_{n+1} + (n+1)a_{n+1} + na_n - \lambda a_n\Big)x^n \\ &= \sum_{n=0}^{+\infty}\Big((n+1)^2 a_{n+1} + (n-\lambda)a_n\Big)x^n \end{aligned}$$

Hence, by the Identity Theorem,

$$f \text{ is a solution of } (E_\lambda) \iff \forall n \in \mathbb{N},\ (n+1)^2 a_{n+1} + (n-\lambda)a_n = 0.$$

2. Hence,

$$f \text{ is a solution of } (E_\lambda) \text{ and } f(0) = 0 \iff \begin{cases} a_0 = 0 \\ \forall n \in \mathbb{N},\ (n+1)^2 a_{n+1} + (n-\lambda)a_n = 0 \end{cases} \iff \forall n \in \mathbb{N},\ a_n = 0.$$

Hence the only solution f of (E_λ) that possesses a power series expansion and such that $f(0) = 0$ is the nil function.

3. a) In the case $\lambda = 1$, f is a solution of (E_{-1}) and $f(0) = 1$ if and only if

$$\forall n \in \mathbb{N},\ a_{n+1} = -\frac{a_n}{n+1} \text{ and } a_0 = 1,$$

i.e., if and only if

$$\forall n \in \mathbb{N},\ a_n = \frac{(-1)^n}{n!}.$$

To see this, we can write, for $n \in \mathbb{N}$,

$$a_{n+1} = a_0 \prod_{k=0}^{n} \frac{a_{k+1}}{a_k} = a_0 \prod_{k=0}^{n} (-1)^k \frac{1}{k+1} = a_0 \frac{(-1)^{n+1}}{(n+1)!},$$

and conclude with $a_0 = f(0) = 1$.
We determine the radius of convergence of f in this case: let $z \in \mathbb{C}^*$. Then,

$$\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| = \frac{|z|}{n+1} \underset{n\to+\infty}{\longrightarrow} 0$$

hence $R = +\infty$.

b) For $x \in \mathbb{R}$,

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} x^n = e^{-x}.$$

Exercise 5.

1. Let $z \in \mathbb{C}^*$ and define $u_n = \dfrac{(-1)^n}{5^n} z^{2n}$. Then

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{z^{2n+2}}{5^{n+1}} \frac{5^n}{z^{2n}}\right| = \frac{|z|^{2n}}{5^n} = \left(\frac{|z|}{\sqrt{5}}\right)^{2n} \underset{n\to+\infty}{\longrightarrow} \begin{cases} +\infty & \text{if } |z| > \sqrt{5} \\ 1 & \text{if } |z| = \sqrt{5} \\ 0 & \text{if } |z| < \sqrt{5}, \end{cases}$$

hence $R = \sqrt{5}$.
Now, for $x \in (-\sqrt{5}, \sqrt{5})$,

$$f(x) = \sum_{n=0}^{+\infty} \left(\frac{-x^2}{5}\right)^n = \frac{1}{1 + x^2/5}.$$

2. Let $z \in \mathbb{C}$. By the Magical Lemma, we know that the series

$$\sum_n \frac{a_n}{n+1} z^n$$

converges if $|z| < R$ and diverges if $|z| > R$. Hence, for $x \in \mathbb{R}$, the series

$$\sum_n \frac{a_n}{n+1} (x^2)^n$$

converges if $|x^2| < R$ and diverges if $|x^2| > R$, i.e., converges if $|x| > \sqrt{R}$ and diverges if $|x| < \sqrt{R}$. Hence $R_g = \sqrt{R}$.
Let $x \in (-R, R)$. Then

$$\int_0^x f(t)\,dt = \sum_{n=0}^{+\infty} \frac{a_n}{n+1} x^{n+1}.$$

Hence, for $x \neq 0$,

$$g(x) = \frac{1}{x^2} \sum_{n=0}^{+\infty} \frac{a_n}{n+1} (x^2)^{n+1} = \frac{1}{x^2} \int_0^{x^2} f(t)\,dt.$$

In the case $x = 0$ we have $g(0) = a_0 = f(0)$.

3. a) Let $z \in \mathbb{C}$ such that $|z| < R_b$. We know that the series $\sum_n b_n z^n$ converges absolutely, hence, since

$$\forall n \in \mathbb{N},\ |a_n z^n| \leq |b_n z^n|,$$

we conclude (by the Comparison Test) that the series $\sum_n a_n z^n$ also converges absolutely, and hence $|z| \leq R_a$. This show that $|z| \leq R_b \implies |z| \leq R_a$, hence $R_b \leq R_a$.

b) i) The radius of convergence of the power series $\sum_n n8^n z^n$ is $1/8$, hence $R \geq \dfrac{1}{8}$.

ii) The radius of convergence of the power series $\sum_n \dfrac{n}{3} z^n$ is 1 and the radius of convergence of the power series $\sum_n n^3 z^n$ is also 1. Hence $R = 1$.

Exercise 6. We first notice that for all $f, g \in E$, $\varphi(f,g) = \varphi(g,f)$, hence we only need to check that φ is linear with respect to one of its arguments.
Let $f, g, h \in E$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} \varphi(f + \lambda g, h) = \int_0^1 (f+\lambda g)'(t)h'(t)\,dt &= \int_0^1 (f'(t) + \lambda g'(t))h'(t)\,dt \\ &= \int_0^1 (f'(t)h'(t)\,dt + \lambda \int_0^1 (g'(t)h'(t)\,dt \\ &= \varphi(f,h) + \lambda\varphi(g,h). \end{aligned}$$

Hence φ is a symmetric bilinear form on E. The quadratic form q associated with φ is:

$$\begin{aligned} q : E &\longrightarrow \mathbb{R} \\ f &\longmapsto \int_0^1 (f'(t))^2\,dt. \end{aligned}$$