INSA | INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — S1 — Math Test #4 — 2h

March 26, 2018

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. The questions of this exercise are independent from each other.

1. For which values of $\alpha > 0$ is the following series convergent?

(1) $$\sum_n \exp\left(\frac{(-1)^n}{n^\alpha}\right) - \cos\left(\frac{(-1)^n}{n^\alpha}\right).$$

2. Is the following series convergent?

(2) $$\sum_n \frac{n!^2}{(2n)!}.$$

3. ~~Let $x \in (-1;1)$.~~ Compute the value of the following sum (you don't need to check that the series converges):

$$S = \sum_{n=1}^{+\infty} \frac{1}{n3^n}.$$

Exercise 2.

1. Show that the numerical series

(S) $$\sum_n \frac{(-1)^n}{1+\sqrt{n}}$$

converges. Is the series (S) absolutely convergent?

We define

$$S = \sum_{n=0}^{+\infty} \frac{(-1)^n}{1+\sqrt{n}}$$

and, for $N \in \mathbb{N}$,

$$S_N = \sum_{n=0}^{N} \frac{(-1)^n}{1+\sqrt{n}} \qquad \text{and} \qquad R_N = \sum_{n=N+1}^{+\infty} \frac{(-1)^n}{1+\sqrt{n}}.$$

2. Let $N \in \mathbb{N}$. Determine the sign of R_N and an upper bound of $|R_N|$.
3. What value of $N \in \mathbb{N}$ will guarantee that S_N is an approximation of S with error less than 10^{-3}?

Exercise 3. Briefly check that the series

$$\sum_n \frac{n^2}{1+n^4}$$

is convergent. Let $\varepsilon > 0$. Find a lower bound on $N \in \mathbb{N}^*$ for

$$S_N = \sum_{n=1}^{N} \frac{n^2}{1+n^4}$$

to be an approximation of

$$S = \sum_{n=1}^{+\infty} \frac{n^2}{1+n^4}$$

with error less than ε.

Exercise 4. Let $\lambda \in \mathbb{R}$. We consider the differential equation

$$(E_\lambda) \qquad xy''(x) + (1+x)y'(x) - \lambda y(x) = 0.$$

1. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that the radius of convergence of the power series $\sum a_n z^n$ is $R > 0$ and define the function

$$\begin{aligned} f \,:\, (-R,R) &\longrightarrow \mathbb{R} \\ x &\longmapsto \sum_{n=0}^{+\infty} a_n x^n. \end{aligned}$$

   Show that f is a solution of Equation (E_λ) if an only if

$$\forall n \in \mathbb{N},\ (n+1)^2 a_{n+1} = (\lambda - n)a_n.$$

2. Deduce the solutions f of Equation (E_λ) that possess a power series expansion and that moreover satisfy $f(0) = 0$.
3. From now on we assume that $\lambda = -1$.
   a) Determine explicitly the coefficients $(a_n)_{n\in\mathbb{N}}$ such that f is a solution of Equation (E_λ) and $f(0) = 1$, as well as the radius of convergence R.
   b) Give an explicit form of these solutions in terms of usual functions.

Exercise 5. The questions of this exercise are independent from each other.

1. Determine the radius of convergence R of the power series

$$f(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{5^n} x^{2n}$$

   and, for $x \in (-R,R)$, determine an explicit expression of $f(x)$.
2. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that the radius of convergence of the power series $\sum_n a_n x^n$ is $R > 0$. We define the function f as

$$\begin{aligned} f \,:\, (-R,R) &\longrightarrow \mathbb{R} \\ x &\longmapsto \sum_{n=0}^{+\infty} a_n x^n. \end{aligned}$$

   Determine the radius of convergence R_g of the power series

$$g(x) = \sum_{n=0}^{+\infty} \frac{a_n}{n+1} x^{2n}$$

   and, for $x \in (-R_g, R_g)$, give an explicit expression of $g(x)$ in terms of f and x.
3. a) Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of complex numbers such that

$$\forall n \in \mathbb{N},\ |a_n| \le |b_n|.$$

   Let R_a be the radius of convergence of the power series $\sum_n a_n z^n$ and let R_b be radius of convergence of the power series $\sum_n b_n z^n$. Prove that $R_b \le R_a$.

   b) What can be said about the radius of convergence R of the power series $\sum_n a_n x^n$ in the following cases?
      i) $\forall n \in \mathbb{N},\ |a_n| \le n8^n$?
      ii) $\forall n \in \mathbb{N},\ \dfrac{n}{3} \le a_n \le n^3$?

Exercise 6. Let $E = C^1(\mathbb{R})$ be the real vector space of all functions of class C^1 on $\mathbb{R}$. Show that the following mapping is a symmetric bilinear form on E:

$$\begin{aligned} \varphi \,:\, E \times E &\longrightarrow \mathbb{R} \\ (f,g) &\longmapsto \int_0^1 f'(t)g'(t)\,\mathrm{d}t. \end{aligned}$$

Give an expression of the quadratic form q associated with φ.