INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Solution of Math Test #5

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Exercise 1.

1. We compute:

$$\varphi(g_0,g_0)=\int_{-1}^{1}\mathrm{d}t=2,$$
$$\varphi(g_0,g_1)=\int_{-1}^{1}t\,\mathrm{d}t=0,$$
$$\varphi(g_1,g_1)=\varphi(g_0,g_2)=\int_{-1}^{1}t^2\,\mathrm{d}t=\frac{2}{3},$$
$$\varphi(g_1,g_2)=\int_{-1}^{1}t^3\,\mathrm{d}t=0,$$
$$\varphi(g_2,g_2)=\int_{-1}^{1}t^4\,\mathrm{d}t=\frac{2}{5}.$$

Hence,

$$M=[\varphi]_{\mathscr{B}}=\begin{pmatrix}2&0&2/3\\0&2/3&0\\2/3&0&2/5\end{pmatrix}.$$

2. We first orthogonalize the basis $\mathscr{B}$:
   - Set $u_0=g_0$.
   - Notice that $u_0\perp g_1$, hence we set $u_1=g_1$.
   - Set $u_2=g_2+\lambda u_0+\mu u_1$, with

$$\lambda=-\frac{\varphi(u_0,g_2)}{\varphi(u_0,u_0)}=-\frac{2/3}{2}=-\frac{1}{3},\qquad \mu=-\frac{\varphi(u_1,g_2)}{\varphi(u_1,u_1)}=0,$$

   i.e.,

$$u_2=g_2-\frac{1}{3}g_0.$$

   Now we divide each of u_0, u_1 and u_2 by their norm to obtain an orthonormal basis: set

$$v_0=\frac{u_0}{\sqrt{\varphi(u_0,u_0)}}=\frac{1}{\sqrt{2}}g_0,$$
$$v_1=\frac{u_1}{\sqrt{\varphi(u_1,u_1)}}=\sqrt{\frac{3}{2}}g_1,$$
$$v_2=\frac{u_2}{\sqrt{\varphi(u_2,u_2)}},$$

   and

$$\varphi(u_2,u_2)=\varphi(g_2,g_2)-\frac{2}{3}\varphi(g_0,g_2)+\frac{1}{9}\varphi(g_0,g_0)=\frac{2}{5}-\frac{4}{9}+\frac{2}{9}=\frac{8}{45},$$

   hence

$$v_2=\frac{3}{2}\sqrt{\frac{5}{2}}g_2-\frac{1}{2}\sqrt{\frac{5}{2}}g_0.$$

   The basis $\mathscr{B}'=(v_0,v_1,v_2)$ is an orthonormal basis of F.

3. a)

$$P=[\mathscr{B}']_{\mathscr{B}}=\begin{pmatrix}1/\sqrt{2}&0&-1/2\times\sqrt{5/2}\\0&\sqrt{3/2}&0\\0&0&3/2\times\sqrt{5/2}\end{pmatrix}.$$

   The matrix P isn't an orthogonal matrix, since clearly ${}^tPP\neq I_3$.
   In fact there's no reason for P to be an orthogonal matrix, since the basis $\mathscr{B}$ isn't an orthonormal basis of F.[2]

   b) $M'=I_3$, since $\mathscr{B}'$ is an orthonormal basis with respect to φ.

   c) $M'={}^tPMP$.

4. a) Let $f\in E$. Since (v_0,v_1,v_2) is an orthonormal basis of F,

$$p_F(f)=\varphi(f,v_0)v_0+\varphi(f,v_1)v_1+\varphi(f,v_2)v_2.$$

   b)

$$p_F(g_3)=\varphi(g_3,v_0)v_0+\varphi(g_3,v_1)v_1+\varphi(g_3,v_2)v_2.$$

   Now,

$$\varphi(g_3,v_0)=0,$$
$$\varphi(g_3,v_1)=\sqrt{\frac{3}{2}}\varphi(g_3,g_1)=2\sqrt{\frac{3}{2}}\frac{1}{5}=\frac{\sqrt{6}}{5},$$
$$\varphi(g_3,v_2)=0.$$

   Hence

$$p_F(g_3)=\frac{\sqrt{6}}{5}v_1=\frac{3}{5}g_1.$$

   c) Observe that

$$m=\inf u\in F\|g_3-u\|_\varphi^2,$$

   and we know that this inf is attained at $p_F(g_3)$:

$$m=\|g_3-p_F(g_3)\|_\varphi^2.$$

   Now,

$$m=\int_{-1}^{1}\left(t^3-\frac{3}{5}t\right)^2\mathrm{d}t=\int_{-1}^{1}\left(t^6-\frac{6}{5}t^4+\frac{9}{25}t^2\right)\mathrm{d}t=\left[\frac{t^7}{7}-\frac{6}{25}t^5+\frac{3}{25}t^3\right]_{t=-1}^{1}=\frac{2}{7}-\frac{12}{25}+\frac{6}{25}=\frac{8}{175}.$$

Exercise 2. Observe that A is a real symmetric matrix, hence A is diagonalizable. 4 is an obvious eigenvalue of A since the matrix

$$A-4I_3=\begin{pmatrix}-4&2&-2\\2&-1&1\\-2&1&-1\end{pmatrix}$$

is clearly of rank 1. So we conclude that 4 is an eigenvalue of A of multiplicity 2. Using the trace, we conclude that -2 is the other eigenvalue of A.
The equation of the eigenspace E_4 is:

$$-2x+y-z=0.$$

Since $E_{-2}\perp E_4$, we conclude that

$$X_{-2}=\begin{pmatrix}-2\\1\\-1\end{pmatrix}$$

is an eigenvector of A associated with the eigenvalue -2.
Now we choose an eigenvector associated with the eigenvalue 4, e.g.,

$$X_4=\begin{pmatrix}0\\1\\1\end{pmatrix}.$$

[2] More precisely, the theorem we covered is the following: if $\mathscr{B}$ is an *orthonormal basis* of a Euclidean vector space (E,φ) and $\mathscr{B}'$ is a family of vectors of E, then

$\mathscr{B}'$ is an orthonormal basis of (E,φ) if and only if the matrix $[\mathscr{B}']_{\mathscr{B}}$ is an orthogonal matrix.

Using a cross-product, we determine another eigenvector associated with the eigenvalue 4:

$$X_4' = X_{-2} \times X_4 = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}$$

We divide these vectors by their norm, and put them in a matrix:

$$P = \begin{pmatrix} -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1\sqrt{2} & 1\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & -1\sqrt{3} \end{pmatrix}.$$

By construction, P is an orthogonal matrix, and if we set

$$D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

we have $A = PD\,{}^tP$.

Exercise 3.

1. Since A is a real symmetric matrix, such a P and D exist.

2.
   - 0 is an obvious eigenvalue since $\operatorname{rk} A = 3 \neq 4$ (the first and last columns are proportional). By the Rank–Nullity Theorem, the multiplicity of 0 is $4-3=1$.
   - -2 is an obvious eigenvalue of A since the matrix

$$A + 2I_4 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

   is of rank 2. This shows that -2 is an eigenvalue of multiplicity $4-2=2$.
   - Another obvious eigenvalue is 2 since the matrix

$$A - 2I_4 = \begin{pmatrix} -3 & 0 & 0 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & -2 & 0 \\ 1 & 0 & 0 & -3 \end{pmatrix}$$

   is of rank 3. This show that 2 is an eigenvalue of multiplicity $4-3=1$.

   (All this is consistent with the trace of A being equal to -2).

3. a) i)

$$q(M) = (a+d)^2 - 4(ad-bc) = a^2 + 2ad + d^2 - 4ad + 4bc = a^2 + d^2 - 2ad + 4bc.$$

   ii) Since q is a homogeneous polynomial of degree 2 with respect to the components of M, we conclude that q is a quadratic form.

   iii) From the form of q thus obtained, we directly determine the matrix of q in the basis $\mathscr{B}$:

$$[q]_{\mathscr{B}} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix},$$

   and we recognize that $[q]_{\mathscr{B}} = -A$. Hence the result with $\alpha = -1$.

   iv) From Question 2, $\operatorname{sign}(A) = (1,2)$. Since $[q]_{\mathscr{B}} = -A$, the signature of q is $\operatorname{sign}(q) = (2,1)$. Hence q is not positive definite, hence the polar form of q is not an inner product on E.

Exercise 4.

1. Let $(x,y) \in \mathbb{R}^2$. Then:

$$\partial_1 f(x,y) = 2xy + 2x, \qquad \partial_2 f(x,y) = x^2 + 3y^2 - 1.$$

Hence,

$$(x,y) \text{ is a critical point of } f \iff \begin{cases} 2xy + 2x = 0 \\ x^2 + 3y^2 - 1 = 0 \end{cases}$$
$$\iff \begin{cases} x = 0 \\ 3y^2 - 1 = 0 \end{cases} \quad \text{or} \quad \begin{cases} y = -1 \\ x^2 + 2 = 0 \end{cases} \ \textit{(impossible)}$$
$$\iff \begin{cases} x = 0 \\ y = \dfrac{1}{\sqrt{3}} \end{cases} \quad \text{or} \quad \begin{cases} x = 0 \\ y = -\dfrac{1}{\sqrt{3}}. \end{cases}$$

Hence f possesses two critical points on $\mathbb{R}^2$, namely

$$\left(0, \frac{1}{\sqrt{3}}\right) \quad \text{and} \quad \left(0, -\frac{1}{\sqrt{3}}\right).$$

To determine the nature of these critical points, we compute the Hessian matrix of f. For $(x,y) \in \mathbb{R}^2$,

$$H_{(x,y)} f = \begin{pmatrix} 2y+2 & 2x \\ 2x & 6y \end{pmatrix}.$$

- For the critical point $(0, 1/\sqrt{3})$:

$$H_{(0,1/\sqrt{3})} f = \begin{pmatrix} 2/\sqrt{3} + 2 & 0 \\ 0 & 6/\sqrt{3} \end{pmatrix},$$

  the signature of which is $(2,0)$, hence f possesses a local minimum at $(0, 1/\sqrt{3})$.
- For the critical point $(0, -1/\sqrt{3})$:

$$H_{(0,1/\sqrt{3})} f = \begin{pmatrix} -2/\sqrt{3} + 2 & 0 \\ 0 & -6/\sqrt{3} \end{pmatrix},$$

  the signature of which is $(1,1)$, hence f possesses a saddle point at $(0, -1/\sqrt{3})$.

a) See Figure 4.

b) Since D is closed and bounded, and since f is continuous, by the Extreme Value Theorem, f is bounded on D and attains its bounds, hence m and M exist.

c) We need to study the minimum and maximum value of f on ∂D:
- first on the lower segment $[-1,1] \times \{0\}$: define the auxiliary function

$$g : \begin{array}{ccc} [-1,1] & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x,0) = x^2. \end{array}$$

  Clearly, $\min g = 0$ and $\max g = 1$.
- on the half circle: define the auxiliary function

$$g : \begin{array}{ccc} [0,\pi] & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(\cos\theta, \sin\theta) = \cos^2\theta. \end{array}$$

  Clearly, $\min g = 0$ and $\max g = 1$.

Now, the only critical point of f in $\mathring{D}$ to consider is $(0, 1/\sqrt{3})$ where f has a local minimum:

$$f\left(0, \frac{1}{\sqrt{3}}\right) = -\frac{2\sqrt{3}}{9}.$$

Finally, we conclude that

$$m = \min_D f = -\frac{2\sqrt{3}}{9}, \qquad \text{and} \qquad M = \max_D f = 1.$$

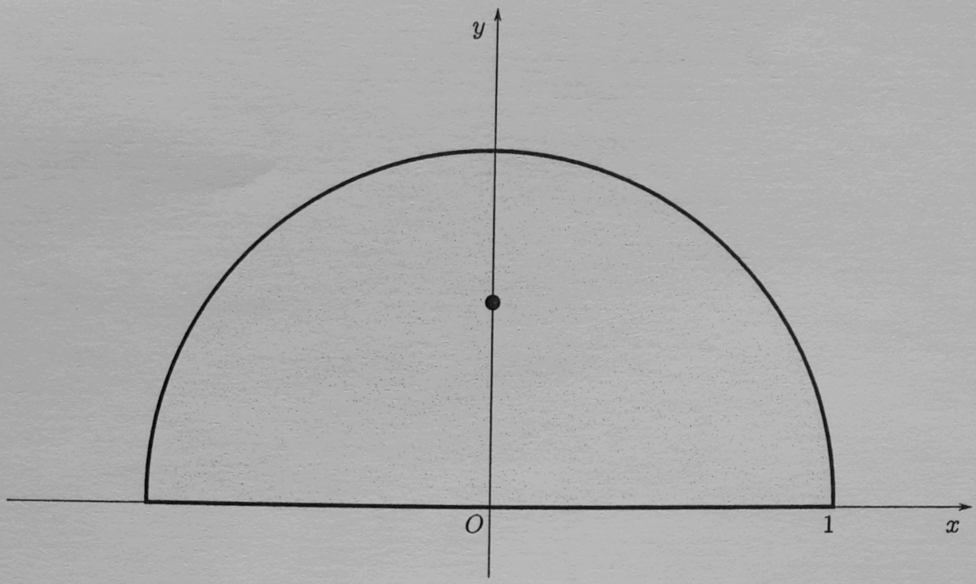

Figure 4. Half disk D of Exercise 4. The dot is the only critical point of f in $\mathring{D}$, at $(0, 1/\sqrt{3})$, where f has a local minimum.

Exercise 5. Since we already know that $F \subset G^{\perp}$, we only need to show that $G^{\perp} \subset F$. Let $u \in G^{\perp}$. Since $E = F + G$, there exists $u_F \in F$ and $u_G \in G$ such that $u = u_F + u_G$. Now, on the one hand, since $u \in G^{\perp}$ and $u_G \in G$,

$$\varphi(u, u_G) = 0.$$

But on the other hand, since $u = u_F + u_G$,

$$\begin{aligned}
\varphi(u, u_G) &= \varphi(u_F + u_G, u_G) \\
&= \varphi(u_F, u_G) + \varphi(u_G, u_G) && \text{since } \varphi \text{ is linear} \\
&= 0 + \varphi(u_G, u_G) && \text{since } u_F \in F \subset G^{\perp} \text{ and } u_G \in G,\ \varphi(u_F, u_G) = 0 \\
&= \|u_G\|_{\varphi}^2,
\end{aligned}$$

where $\|.\|_{\varphi}$ is the norm associated with φ (this is valid since φ is an inner product). Hence $\|u_G\|_{\varphi}^2 = 0$, and we conclude that $u_G = 0_E$. Hence $u = u_F \in F$. Conclusion:

$$\forall u \in G^{\perp},\ u \in F,$$

which means that $G^{\perp} \subset F$. Hence $F = G^{\perp}$.