INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — S2 — Math Test #5 — 2h

June 13, 2018

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. Let $E = C\big([-1,1],\mathbb{R}\big)$ be the vector space of real-valued continuous functions on $[-1,1]$, and define

$$\begin{aligned}\varphi : E\times E &\longrightarrow \mathbb{R}\\ (f,g) &\longmapsto \int_{-1}^{1} f(t)g(t)\,\mathrm{d}t.\end{aligned}$$

You're given that φ is an inner product on E.
For $n \in \mathbb{N}$ we define the element g_n of E as:

$$\begin{aligned}g_n : [-1,1] &\longrightarrow \mathbb{R}\\ t &\longmapsto t^n.\end{aligned}$$

We define

$$\mathscr{B} = (g_0, g_1, g_2).$$

and $F = \operatorname{Span}\{g_0, g_1, g_2\}$. You're given that $\mathscr{B}$ is a basis of F (it's quite obvious that $\mathscr{B}$ is an independent family).
We still denote by φ the restriction of φ to F.

1. Explicit the matrix $M = [\varphi]_{\mathscr{B}}$ of φ in the basis $\mathscr{B}$.
2. Apply the Gram-Schmidt process to the basis $\mathscr{B}$ of F to determine an orthonormal basis $\mathscr{B}'$ of F. *You will call v_0, v_1 and v_2 the vectors of $\mathscr{B}'$.*
3. a) Write the matrix $P = [\mathscr{B}']_{\mathscr{B}}$. Is P an orthogonal matrix? (Is this surprising?)
   b) Give (without any computations) the matrix $M' = [\varphi]_{\mathscr{B}'}$ of φ in $\mathscr{B}'$.
   c) What relation exists between M, M' and P? *(no justifications required).*
4. Let $p_F : E \longrightarrow E$ be the orthogonal projection onto F (orthogonal with respect to φ).
   a) For $f \in E$, recall the general formula for $p_F(f)$.
   b) Determine $p_F(g_3)$.
   c) Determine the value of

$$m = \inf_{(a,b,c)\in\mathbb{R}^3} \int_{-1}^{1} \left(t^3 - a - bt - ct^2\right)^2 \mathrm{d}t.$$

Exercise 2. Let

$$A = \begin{pmatrix} 0 & 2 & -2\\ 2 & 3 & 1\\ -2 & 1 & 3\end{pmatrix}.$$

Find an orthogonal matrix P and a diagonal matrix D such that $A = PD\,{}^tP$.

Exercise 3. We define the matrix

$$A = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

1. Explain why there exists a diagonal matrix D and an orthogonal matrix P such that $A = PD\,{}^tP$. *(You're not asked to determine P and D explicitly).*

2. Determine the eigenvalues of A as well as their multiplicities.

3. Let $E = M_2(\mathbb{R})$ be the real vector space of 2×2 matrices with real coefficients. We define

$$\begin{aligned} q : E &\longrightarrow \mathbb{R} \\ M &\longmapsto (\mathrm{tr}(M))^2 - 4\det(M). \end{aligned}$$

   We also define the following vectors of E:

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

   and $\mathscr{B} = (A_1, A_2, A_3, A_4)$. You're given that $\mathscr{B}$ is a basis of E.

   a) i) Let $M \in E$, say $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for some $a, b, c, d \in \mathbb{R}$. Give an explicit expression of $q(M)$ in terms of a, b, c, d.
      ii) Deduce that q is a quadratic form on E.
      iii) Check that there exists $\alpha \in \mathbb{R}$ such that $[q]_{\mathscr{B}} = \alpha A$ (and give the value of α).

   b) Give the signature of q. Is the polar form of q an inner product on E?

Exercise 4. Let

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto x^2 - y + y^3 + yx^2, \end{aligned}$$

and

$$D = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 \le 1,\ y \ge 0\right\}.$$

1. Find the critical points of f on $\mathbb{R}^2$, and determine their nature.

2. a) On the same figure, sketch D and the critical points of f that belong to D.
   b) Explain why $m = \min\limits_D f$ and $M = \max\limits_D f$ exist.
   c) Determine the value of $m = \min\limits_D f$ and of $M = \max\limits_D f$.

Exercise 5. Let E be a real vector space and let φ be an inner product on G. Let F and G be two subspaces of E such that:

- $E = F + G$,
- $F \subset G^\perp$.

Show that $F = G^\perp$.
You may want to proceed as follows: let $u \in G^\perp$. Explain why there exists $u_F \in F$ and $u_G \in G$ such that $u = u_F + u_G$. Then consider $\varphi(u, u_G)$...