INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Solution of Math Test #1

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October 22, 2018

Exercise 1. The statement is false: define

$$f\ :\ \begin{array}{rcl}[1,+\infty) & \longrightarrow & \mathbb{R}\\ x & \longmapsto & \dfrac{1}{x}.\end{array}$$

Clearly f is continuous and $\lim\limits_{x\to+\infty} f(x)=0$. Yet it is well-known that the improper integral

$$\int_1^{+\infty} f(x)\,\mathrm{d}x$$

diverges (Riemann integral at $+\infty$ with $\alpha = 1 \leq 1$).

Exercise 2.

1. The function $x \mapsto \mathrm{e}^{-\sqrt{x^2+x}}$ is continuous on $[1,+\infty)$ hence the improper integral (1) is improper at $+\infty$. Now observe that

$$\forall x\in[1,+\infty),\ x^2+x\geq x^2$$

hence

$$\forall x\in[1,+\infty),\ -\sqrt{x^2+x}\leq -x$$

hence

$$\forall x\in[1,+\infty),\ 0\leq \mathrm{e}^{-\sqrt{x^2+x}}\leq \mathrm{e}^{-x}.$$

Now we know that the improper integral

$$\int_1^{+\infty}\mathrm{e}^{-x}\,\mathrm{d}x$$

is convergent hence, by the comparison test, the improper integral (1) is convergent.

2. The function

$$t\mapsto \frac{1-\cos(t)}{t^2}\mathrm{e}^{-t}$$

is continuous on $(0,+\infty)$ hence the improper integral (2) is improper at 0^+ and at $+\infty$.

   - Convergence at 0^+: by the well-known equivalents,

   $$\frac{1-\cos(t)}{t^2}\mathrm{e}^{-t}\underset{x\to0^+}{\sim}\frac12\times1=\frac12\underset{x\to0^+}{\longrightarrow}\frac12.$$

   Hence the improper integral (2) is falsely improper at 0^+ hence the improper integral (2) converges at 0^+.

   - Convergence at $+\infty$: observe that

   $$\forall t\in[1,+\infty),\ 0\leq\frac{1-\cos(t)}{t^2}\mathrm{e}^{-t}\leq\frac{2}{t^2}.$$

   We know that the improper integral

   $$\int_1^{+\infty}\frac{2}{t^2}\,\mathrm{d}t$$

   converges at $+\infty$ (Riemann at $+\infty$ with $\alpha=2>1$) hence, by the comparison test, the improper integral (2) converges at $+\infty$.

   We hence conclude that the improper integral (2) is convergent.

3. Let $x \in \mathbb{R}$. The function

$$t \mapsto \frac{t^x}{1+t}$$

is continuous on $(0,1]$ (this interval must be opened at 0 to take care of the case $x<0$), hence the improper integral (3) is improper at 0^+. Now,

$$\frac{t^x}{1+t} \underset{t\to 0^+}{\sim} t^x = \frac{1}{t^{-x}} > 0,$$

and we know, by Riemann at $+\infty$, that the improper integral

$$\int_0^1 \frac{\mathrm{d}t}{t^{-x}}$$

converges if and only if $\alpha = -x < 1$. Hence, by the equivalent test, the improper integral (3) converges if and only if $x > -1$.

Exercise 3.

1. The function

$$t \mapsto \frac{\ln(t)}{1+t^2}$$

is continuous on $[1,+\infty)$ hence the improper integral I is improper at $+\infty$. Now,

$$t^{3/2}\frac{\ln(t)}{1+t^2} \underset{t\to+\infty}{\sim} t^{3/2}\frac{\ln(t)}{t^2} = \frac{\ln(t)}{\sqrt{t}} \underset{t\to+\infty}{\longrightarrow} 0.$$

Hence there exists $A > 1$ such that

$$\forall t \in [A,+\infty),\ t^{3/2}\frac{\ln(t)}{1+t^2} \leq 1.$$

Then

$$\forall t \in [A,+\infty),\ 0 \leq \frac{\ln(t)}{1+t^2} \leq \frac{1}{t^{3/2}}.$$

Now we know that the improper integral

$$\int_A^{+\infty} \frac{\mathrm{d}t}{t^{3/2}}$$

is convergent at $+\infty$ (Riemann at $+\infty$ with $\alpha = 3/2 > 1$) hence, by the comparison test, the improper integral I converges at $+\infty$.

2. The function

$$t \mapsto \frac{\ln(t)}{1+t^2}$$

is continuous on $(0,1]$ hence the improper integral J is improper at 0^+. Let $X \in (0,1)$. Then, using the substitution $s = 1/t$ yields:

$$\begin{aligned}
\int_X^1 \frac{\ln(t)}{1+t^2}\,\mathrm{d}t &= \int_{1/X}^1 \frac{\ln(1/s)}{1+1/s^2}\left(-\frac{\mathrm{d}s}{s^2}\right) \\
&= \int_{1/X}^1 \frac{-\ln(s)}{s^2+1}(-\mathrm{d}s) \\
&= \int_{1/X}^1 \frac{\ln(s)}{s^2+1}\mathrm{d}s \\
&= -\int_1^{1/X} \frac{\ln(s)}{s^2+1}\mathrm{d}s \\
&\underset{X\to 0^+}{\longrightarrow} \int_1^{+\infty} \frac{\ln(s)}{1+s^2}\,\mathrm{d}s = -I.
\end{aligned}$$

Hence J is convergent and $J = -I$.

3. Let $t \in [0,1]$. We know (from the sum of the terms of a geometric progression of ratio $-t^2 \neq 1$) that

$$\sum_{k=0}^{n}(-1)^k t^{2k} = \sum_{k=0}^{n}\left(-t^2\right)^k = \frac{1-\left(-t^2\right)^{n+1}}{1-\left(-t^2\right)} = \frac{1}{1+t^2} - \frac{(-1)^{n+1}t^{2n+2}}{1+t^2}.$$

Hence

$$\frac{1}{1+t^2} = \sum_{k=0}^{n}(-1)^k t^{2k} + \frac{(-1)^{n+1}t^{2n+2}}{1+t^2}.$$

4. Let $k \in \mathbb{N}$. The function $t \mapsto t^{2k}\ln(t)$ is continuous on $(0,1]$, hence the improper integral U_k is improper at 0^+. Let $X \in (0,1)$. By an integration by parts,

$$\begin{aligned}\int_X^1 t^{2k}\ln(t)\,\mathrm{d}t &= \left[\frac{t^{2k+1}}{2k+1}\ln(t)\right]_{t=X}^{t=1} - \int_X^1 \frac{t^{2k+1}}{2k+1}\frac{1}{t}\,\mathrm{d}t \\ &= -\frac{X^{2k+1}}{2k+1}\ln(X) - \int_X^1 \frac{t^{2k}}{2k+1}\,\mathrm{d}t \\ &= -\frac{X^{2k+1}}{2k+1}\ln(X) - \left[\frac{t^{2k+1}}{(2k+1)^2}\right]_{t=X}^{t=1} \\ &= -\frac{X^{2k+1}}{2k+1}\ln(X) - \frac{1}{(2k+1)^2} + \frac{X^{2k+1}}{(2k+1)^2} \\ &\underset{X\to 0^+}{\longrightarrow} -\frac{1}{(2k+1)^2}.\end{aligned}$$

Hence the improper integral U_k converges and $U_k = -\dfrac{1}{(2k+1)^2}$.

5. Let $k \in \mathbb{N}$. Observe that

$$\forall t \in (0,1],\ 0 \geq \frac{t^{2k+2}\ln(t)}{1+t^2} \geq t^{2k+2}\ln(t).$$

Hence, by the comparison test we conclude that the improper integral

$$\int_0^1 \frac{t^{2k+2}\ln(t)}{1+t^2}\,\mathrm{d}t$$

is convergent and that

$$0 \geq \int_0^1 \frac{t^{2k+2}\ln(t)}{1+t^2}\,\mathrm{d}t \geq \int_0^1 t^{2k+2}\ln(t)\,\mathrm{d}t = -\frac{1}{(2k+3)^2}.$$

Hence

$$0 \leq \left|\int_0^1 \frac{t^{2k+2}\ln(t)}{1+t^2}\,\mathrm{d}t\right| \leq \frac{1}{(2k+3)^2}.$$

Let $n \in \mathbb{N}$. By Question 3, we have

$$\forall t \in [0,1],\ \frac{1}{1+t^2} = \sum_{k=0}^{n}(-1)^k t^{2k} + \frac{(-1)^{n+1}t^{2n+2}}{1+t^2},$$

hence

$$\forall t \in [0,1],\ \frac{\ln(t)}{1+t^2} = \sum_{k=0}^{n}(-1)^k t^{2k}\ln(t) + \frac{(-1)^{n+1}t^{2n+2}\ln(t)}{1+t^2}.$$

Now since all the improper integrals

$$\int_0^1 t^{2k}\ln(t)\,\mathrm{d}t,\ k \in \mathbb{N}, \qquad \text{and} \qquad \int_0^1 \frac{t^{2n+2}\ln(t)}{1+t^2}\,\mathrm{d}t$$

are convergent, we conclude that

$$\forall n \in \mathbb{N},\ J = \sum_{k=0}^{n}\int_0^1 (-1)^k t^{2k}\ln(t)\,\mathrm{d}t + \int_0^1 \frac{(-1)^{n+1}t^{2n+2}\ln(t)}{1+t^2}\,\mathrm{d}t$$

$$= \sum_{k=0}^{n} (-1)^{k+1} \frac{1}{(2k+1)^2} \; + \; \int_0^1 \frac{(-1)^{n+1} t^{2n+2} \ln(t)}{1+t^2} \, \mathrm{d}t.$$

Now, by the inequality obtained in the first part of this question (and the Squeeze Theorem),

$$\lim_{n \to +\infty} \int_0^1 \frac{(-1)^{n+1} t^{2n+2} \ln(t)}{1+t^2} \, \mathrm{d}t = 0.$$

Hence,

$$J = \lim_{n \to +\infty} \sum_{k=0}^{n} (-1)^{k+1} \frac{1}{(2k+1)^2}.$$

Remembering that $I = -J$ yields

$$I = \lim_{n \to +\infty} \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)^2}.$$

Exercise 4.

1. To plot the unit ball of N we separate several cases. Let $(x, y) \in E$ such that $x \geq 0$, $y \geq 0$ and $y \leq x$. Then

$$(x, y) \in \overline{B} \iff N\big((x, y)\big) \leq 1 \iff x + y + x \leq 1 \iff y \leq 1 - 2x.$$

From this inequality we obtain the part of $\overline{B}$ that lies in $\big\{(x, y) \in E \bigm| x \geq 0,\; y \geq 0,\; y \leq x\big\}$.

Now we observe that

$$\forall (x, y) \in E,\; N\big((y, x)\big) = N\big((x, y)\big),$$

from which we conclude that $\overline{B}$ is symmetric with respect to the line $y = x$. At this point we obtain the part of $\overline{B}$ that lies in the first quadrant $\big\{(x, y) \in E \bigm| x \geq 0,\; y \geq 0\big\}$.

Now we observe that

$$\forall (x, y) \in E,\; N\big((-x, y)\big) = N\big((x, y)\big),$$

from which we conclude that $\overline{B}$ is symmetric with respect to the y-axis. At this point we obtain the part of $\overline{B}$ that lies in the upper half plane $\big\{(x, y) \in E \bigm| y \geq 0\big\}$.

Finally we observe that

$$\forall (x, y) \in E,\; N\big((x, -y)\big) = N\big((x, y)\big),$$

from which we conclude that $\overline{B}$ is symmetric with respect to the x-axis. At this point we obtain $\overline{B}$.

The ball $\overline{B}$ is represented in Figure 1.

2. a) The vector space E is a finite-dimensional vector space, hence all the norms on E are equivalent. Hence N and $\|\cdot\|_2$ are equivalent.

   b) Let $u \in E$ such that $u \neq 0_E$.

   - Since

$$\left\| \frac{u}{\sqrt{5}\|u\|_2} \right\|_2 = \frac{1}{\sqrt{5}}$$

     we conclude that

$$\frac{u}{\sqrt{5}\|u\|_2} \in \overline{B_2\left(\frac{1}{\sqrt{5}}\right)},$$

     and hence, by the first inclusion given, that

$$\frac{u}{\sqrt{5}\|u\|_2} \in \overline{B}$$

     and hence

$$N\left(\frac{u}{\sqrt{5}\|u\|_2}\right) \leq 1$$

     and hence

$$N(u) \leq \sqrt{5}\|u\|_2.$$

     We conclude that $\beta = \sqrt{5}$ fulfills the condition.

- Similarly,

$$N\left(\frac{u}{N(u)}\right) = 1$$

hence

$$\frac{u}{N(u)} \in \overline{B}$$

hence, by the second inclusion given,

$$\frac{u}{N(u)} \in \overline{B_2\left(\frac{1}{2}\right)}$$

hence

$$\left\|\frac{u}{N(u)}\right\|_2 \leq \frac{1}{2},$$

hence

$$\|u\|_2 \leq \frac{1}{2}N(u).$$

We conclude that $\alpha = 1/2$ fulfills the condition.

Exercise 5.

1. - Positive homogeneity: let $P \in E$ and $\lambda \in \mathbb{R}$. Then

$$N(\lambda P) = \int_0^1 \left|(1-t)\lambda P(t)\right| \mathrm{d}t = |\lambda| \int_0^1 \left|(1-t)P(t)\right| \mathrm{d}t = |\lambda| N(P).$$

   - Triangle inequality: let $P, Q \in E$. Then (since $0 < 1$),

$$N(P+Q) = \int_0^1 \left|(1-t)\big(P(t)+Q(t)\big)\right| \mathrm{d}t \leq \int_0^1 \Big(\left|(1-t)P(t)\right| + \left|(1-t)Q(t)\right|\Big) \mathrm{d}t = N(P) + N(Q).$$

   - Let $P \in E$ such that $N(P) = 0$. Then

$$\int_0^1 \left|(1-t)P(t)\right| \mathrm{d}t = 0.$$

Since the function $t \mapsto \left|(1-t)P(t)\right|$ is *non-negative and continuous* on $[0,1]$, we conclude that

$$\forall t \in [0,1],\ (1-t)P(t) = 0.$$

Hence,

$$\forall t \in [0,1),\ P(t) = 0.$$

We notice that the polynomial P has an infinite number of roots, which can only happen for $P = 0_E$.

Hence N is a norm on E.

2. - Distance between 1 and P_1:

$$N(1-P_1) = \int_0^1 \left|(1-t)t\right| \mathrm{d}t = \int_0^1 \left(t - t^2\right) \mathrm{d}t = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

   - Distance between 1 and P_2:

$$N(1-P_2) = \int_0^1 \left|(1-t)(1-t)\right| \mathrm{d}t = \int_0^1 (1-t)^2 \,\mathrm{d}t = \frac{1}{3}.$$

Hence P_1 is closer to 1 than P_2.

3. a) Let $n \in \mathbb{N}$. Then

$$N(P_n - 0_E) = \int_0^1 \left|(1-t)t^n\right| \mathrm{d}t = \int_0^1 \left(t^n - t^{n+1}\right) \mathrm{d}t = \frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{(n+1)(n+2)} \underset{n \to +\infty}{\longrightarrow} 0.$$

Hence the sequence $(P_n)_{n \in \mathbb{N}}$ converges to 0_E with respect to N.

b) Let $n \in \mathbb{N}$. Then

$$N(Q_n - 0_E) = N(nP_n) = nN(P_n) = \frac{n}{(n+1)(n+2)} \underset{n\to+\infty}{\longrightarrow} 0.$$

Hence the sequence $(Q_n)_{n\in\mathbb{N}}$ converges to 0_E with respect to N.

4. a) Let $n \in \mathbb{N}$. Then

$$f(P_n) = 1.$$

b) Since $(P_n)_{n\in\mathbb{N}}$ converges to 0_E for N, if f were continuous we would have

$$\lim_{n\to+\infty} \left|f(P_n) - f(0_E)\right| = 0.$$

Now for $n \in \mathbb{N}$, (since $f(0_E) = 0$),

$$\left|f(P_n) - f(0_E)\right| = \left|f(P_n)\right| = P_n(1) = 1 \underset{n\to+\infty}{\longrightarrow} 1 \neq 0.$$

We conclude that f is not continuous (with respect to N) at 0_E.

5. Notice that for $n \in \mathbb{N}^*$, $g(Q_n) = nP_n' = n^2 P_{n-1}$. Since $(Q_n)_{n\in\mathbb{N}}$ converges to 0_E for the norm N, if g were continuous at 0_E we would have

$$\lim_{n\to+\infty} N\big(g(Q_n) - g(0_E)\big) = 0.$$

But for $n \in \mathbb{N}^*$, (since $g(0_E) = 0_E$),

$$N\big(g(Q_n) - g(0_E)\big) = N\big(g(Q_n)\big) = N\big(n^2 P_{n-1}\big) = n^2 N(P_{n-1}) = \frac{n^2}{n(n+1)} \underset{n\to+\infty}{\longrightarrow} 1 \neq 0.$$

Hence g is not continuous at 0_E.

Exercise 6. In this exercise we use the 2-norm on $\mathbb{R}^2$ (all the norms on $\mathbb{R}^2$ being equivalent, we can use the one we want).

1. Case $\alpha + \beta > 2$: let $(x, y) \in U$. Then, remembering the following useful inequalities

$$|x| \leq \|(x,y)\|_2 \qquad \text{and} \qquad |y| \leq \|(x,y)\|_2$$

we obtain:

$$\left|f(x,y) - 0\right| = \frac{|x|^\alpha |y|^\beta}{\left\|(x,y)\right\|_2} \leq \frac{\left\|(x,y)\right\|_2^{\alpha+\beta}}{\left\|(x,y)\right\|_2} = \left\|(x,y)\right\|_2^{\alpha+\beta-2} \underset{\|(x,y)\|_2\to 0}{\longrightarrow} 0,$$

since $\alpha + \beta - 2 > 0$. Hence $\lim\limits_{(x,y)\to 0} f(x,y) = 0$.

2.
   - Case $\alpha + \beta = 2$: we show that the directional limits of f are not equal: Let $t \in \mathbb{R}^*$. Then

$$f(t,t) = \frac{|t|^{\alpha+\beta}}{2t^2} = \frac{1}{2}|t|^{\alpha+\beta-2} = \frac{1}{2} \underset{t\to 0}{\longrightarrow} \frac{1}{2},$$

   and

$$f(t,0) = 0 \underset{t\to 0}{\longrightarrow} 0.$$

   Since $(t,t) \underset{t\to 0}{\longrightarrow} (0,0)$ and $(t,0) \underset{t\to 0}{\longrightarrow} (0,0)$, and since both limits we computed aren't equal, we conclude that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ doesn't exist (composition of limits theorem, together with the uniqueness of limits property).
   - Case $\alpha + \beta < 2$: for $t \in \mathbb{R}^*$,

$$f(t,t) = |t|^{\alpha+\beta-2} \underset{t\to 0}{\longrightarrow} +\infty.$$

   Hence, since $(t,t) \underset{t\to 0}{\longrightarrow} (0,0)$, we conclude that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ doesn't exist (in $\mathbb{R}$).

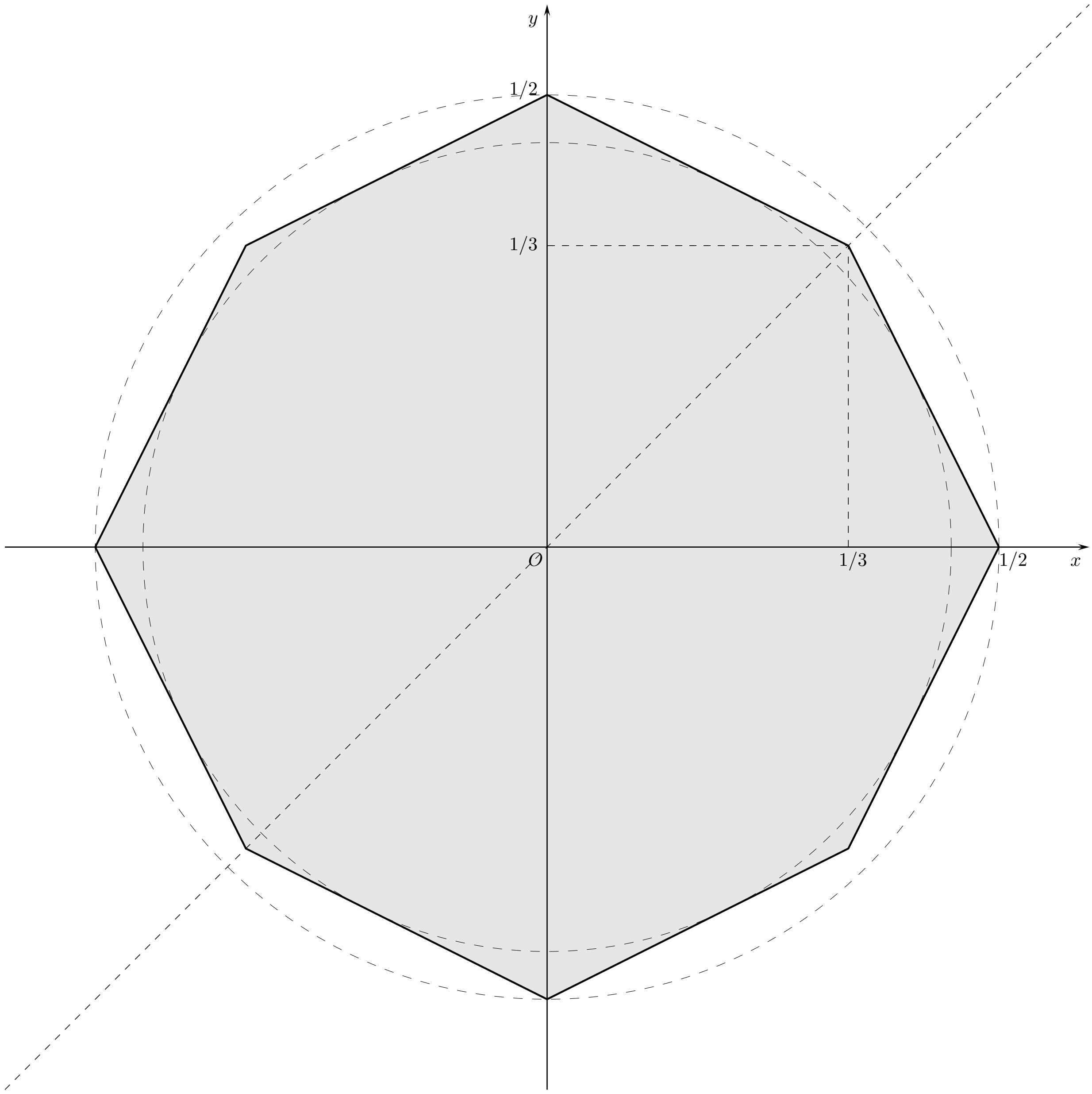

Figure 1. Unit ball $\overline{B}$ of Exercise 4. We have also represented (in dashed) the boundaries of $\overline{B_2(1/2)}$ and $\overline{B_2(1/\sqrt{5})}$, and the straight line of equation $y = x$.