INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — S1 — Math Test #1 — 2h

October 22, 2018

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

The marks are given as a guide only; the final marking scheme might differ *slightly* from the marks provided here.

Exercise 1 (*1 mark*). Is the following proposition true or false? if it is true, prove it; if it is false, provide a counter-example.

Let $f : [1, +\infty) \to \mathbb{R}$ be a continuous function. If $\lim_{x \to +\infty} f(x) = 0$ then the improper integral $\int_1^{+\infty} f(x)\,\mathrm{d}x$ converges.

Exercise 2 (*3 marks*). Determine the nature of the following improper integrals:

(1) $\displaystyle\int_1^{+\infty} \mathrm{e}^{-\sqrt{x^2+x}}\,\mathrm{d}x$, (2) $\displaystyle\int_0^{+\infty} \frac{1-\cos(t)}{t^2}\mathrm{e}^{-t}\,\mathrm{d}t$, (3) for $x \in \mathbb{R}$, $\displaystyle\int_0^1 \frac{t^x}{1+t}\,\mathrm{d}t$.

Exercise 3 (*5 marks*). The goal of this exercise is to give an expression of the value of the following improper integral (after showing that it converges):

$$I = \int_1^{+\infty} \frac{\ln(t)}{1+t^2}\,\mathrm{d}t.$$

1. Show that the improper integral I is convergent.
2. Use a substitution to show that the improper integral
$$J = \int_0^1 \frac{\ln(t)}{1+t^2}\,\mathrm{d}t$$
is convergent, and that $J = -I$.
3. Show that
$$\forall n \in \mathbb{N}, \quad \forall t \in [0,1], \quad \frac{1}{1+t^2} = \sum_{k=0}^{n} (-1)^k t^{2k} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}.$$
4. Let $k \in \mathbb{N}$. Show that the improper integral
$$U_k = \int_0^1 t^{2k} \ln(t)\,\mathrm{d}t$$
is convergent, and that
$$\int_0^1 t^{2k} \ln(t)\,\mathrm{d}t = -\frac{1}{(2k+1)^2}.$$
5. Deduce that
$$\forall k \in \mathbb{N}, \quad \left| \int_0^1 \frac{t^{2k+2}\ln(t)}{1+t^2}\,\mathrm{d}t \right| \le \frac{1}{(2k+3)^2}.$$
Deduce that
$$I = \lim_{n \to +\infty} \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)^2}.$$

Taylor expansia?

$\frac{1}{1+x} = ?$

$f(a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)\, x^k}{k!} +$

$\left(\frac{1}{1+x}\right)' = -\frac{1}{(1+x)^2}$

$\frac{1}{1+x} = 1 +$

Exercise 4 (*2.5 marks*). Let $E = \mathbb{R}^2$ and define

$$\begin{array}{rccl} N: & E & \longrightarrow & \mathbb{R}_+ \\ & (x,y) & \longmapsto & |x| + |y| + \max\{|x|,|y|\}. \end{array}$$

You're given that N is a norm on E, and you don't need to justify this fact.

1. Plot the closed unit ball $\overline{B}$ of N.

   You may want to use the symmetries of N: first determine the part of $\overline{B}$ that lies in $\{(x,y) \in E \mid x \geq 0,\ y \geq 0,\ y \leq x\}$. Then use the symmetries of N to get the other parts of $\overline{B}$: $\forall (x,y) \in E,\ N(y,x) = N(x,-y) = N(-x,y) = N(x,y)$.

2. a) Explain (as concisely as possible) why the norms N and $\|\cdot\|_2$ are equivalent.

   b) For $r > 0$ we denote by $\overline{B_2(r)}$ the closed ball for the 2-norm, centered at 0_E of radius r. You're given that

   $$\overline{B_2\left(\frac{1}{\sqrt{5}}\right)} \subset \overline{B} \subset \overline{B_2\left(\frac{1}{2}\right)}.$$

   Deduce values of $\alpha, \beta \in \mathbb{R}_+^*$ such that

   $$\alpha\|\cdot\|_2 \leq N \leq \beta\|\cdot\|_2.$$

Exercise 5 (*6 marks*). Let $E = \mathbb{R}[X]$ be the real vector space of polynomials with indeterminate X and real coefficients. We define

$$\begin{array}{rcl} N: E & \longrightarrow & \mathbb{R}_+ \\ P & \longmapsto & \displaystyle\int_0^1 \left|(1-t)P(t)\right| \mathrm{d}t. \end{array}$$

1. Prove that N is a norm on E.

2. *(Just a random question that will not be of any use in the sequel).* Define the following elements of E:

   $$P_1 = X, \qquad\qquad P_2 = (1-X).$$

   Which of P_1 or P_2 is the closest (with respect to N) to 1?

3. We define the sequences $(P_n)_{n\in\mathbb{N}}$ and $(Q_n)_{n\in\mathbb{N}}$ of elements of E as

   $$\forall n \in \mathbb{N}, \quad P_n = X^n, \qquad Q_n = nP_n.$$

   a) Show that the sequence $(P_n)_{n\in\mathbb{N}}$ converges to 0_E for the norm N.

   b) Show that the sequence $(Q_n)_{n\in\mathbb{N}}$ converges to 0_E for the norm N.

4. Define the mapping

   $$\begin{array}{rcl} f: E & \longrightarrow & \mathbb{R} \\ P & \longmapsto & P(1). \end{array}$$

   a) For $n \in \mathbb{N}$, compute the value of $f(P_n)$, where $P_n = X^n$.

   b) Is f continuous (with respect to N) at 0_E?

5. Define the mapping

   $$\begin{array}{rcl} g: E & \longrightarrow & E \\ P & \longmapsto & P'. \end{array}$$

   Is g continuous (with respect to N) at 0_E?

Exercise 6 (*2.5 marks*). Let $U = \mathbb{R}^2 \setminus \{(0,0)\}$. Let $\alpha, \beta \in \mathbb{R}_+^*$ and define

$$\begin{array}{rccl} f: & U & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & \dfrac{|x|^\alpha |y|^\beta}{x^2+y^2}. \end{array}$$

1. Show that if $\alpha + \beta > 2$ then $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

2. Show that if $\alpha + \beta \leq 2$ then the limit $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist (in $\mathbb{R}$). *You may want to consider the cases $\alpha+\beta = 2$ and $\alpha + \beta < 2$ separately.*