INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Solution of Math Test #2

Romaric Pujol, romaric.pujol@insa-lyon.fr

December 10, 2018

Exercise 1.

1. The function

$$p:\ \begin{array}{rcl}\mathbb{R}^2 & \longrightarrow & \mathbb{R}\\ (x,y) & \longmapsto & y\end{array}$$

is obviously continuous, and $(0,+\infty)$ is an open subset of $\mathbb{R}$, hence $p^{[-1]}\big((0,+\infty)\big)$ is an open subset of $\mathbb{R}^2$. The conclusion follows by observing that $D=p^{[-1]}\big((0,+\infty)\big)$.

2. a) Let $y\in\mathbb{R}_+^*$. Then

$$y-\sqrt{y}=\frac{\big(y-\sqrt{y}\big)\big(y+\sqrt{y}\big)}{y+\sqrt{y}}=\frac{y^2-y}{y+\sqrt{y}}=(y-1)\frac{y}{y+\sqrt{y}}$$

hence

$$|y-\sqrt{y}|=|y-1|\frac{y}{y+\sqrt{y}}\leq|y-1|.$$

b) Let $(x,y)\in D$. Then

$$|y-\sqrt{y}|\leq|y-1|\leq\big\|(x,y-1)\big\|_2,$$

since we know that for all $(X,Y)\in\mathbb{R}^2$, $|Y|\leq\big\|(X,Y)\big\|_2$.

c) Let $(x,y)\in D$. Then:

$$\begin{aligned}\big|f(x,y)\big| &= \big|y-\sqrt{y}\big|\Big|\ln\big(x^2+(y-1)^2\big)\Big|\\ &= \big|y-\sqrt{y}\big|\Big|\ln\Big(\big\|(x,y-1)\big\|_2^2\Big)\Big|\\ &\leq \big\|(x,y-1)\big\|_2\Big|\ln\Big(\big\|(x,y-1)\big\|_2^2\Big)\Big| \xrightarrow[\|(x,y-1)\|_2\to0]{} 0\end{aligned}$$

hence f is continuous at $(0,1)$.

3. Let $(x,y)\in D\setminus\big\{(0,1)\big\}$.

a)

$$\begin{aligned}\partial_1 f(x,y) &= \frac{2\big(y-\sqrt{y}\big)}{x^2+(y-1)^2}\\ \partial_2 f(x,y) &= \big(1-\frac{1}{2\sqrt{y}}\big)\ln\big(x^2+(y-1)^2\big)+2\frac{(y-\sqrt{y})(y-1)}{x^2+(y-1)^2}.\end{aligned}$$

b) The first-order partial derivatives of f are clearly continuous on $D\setminus\big\{(0,1)\big\}$ (as they are obtained by elementary operations and continuous functions only), hence f is of class C^1 on $D\setminus\big\{(0,1)\big\}$.

c) First evaluate f and its first order partial derivatives at $(2,1)$:

$$f(2,1)=0,\partial_1 f(2,1) \qquad =0,\partial_2 f(2,1) \qquad =\ln(2)$$

hence the first order Taylor–Young expansion of f at $(2,1)$ is

$$f(2+h_x,1+h_y)\underset{(h_x,h_y)\to(0,0)}{=}h_y\ln(2)+o\big(\|(h_x,h_y)\|_2\big).$$

or, equivalently

$$f(x,y)\underset{(x,y)\to(2,1)}{=}(y-1)\ln(2)+o\big(\|((x-2,y-1)\|_2\big).$$

d) Since f is of class C^1 on D, f is differentiable at $(2,1)$, and

$$\nabla_{(1,1)}f(2,1)=\mathrm{d}_{(2,1)}f(1,1)=\partial_1 f(2,1)+\partial_2 f(2,1)=\ln(2).$$

4. a) Let $h \in \mathbb{R}^*$. Then

$$\frac{f(x,1)-f(0,1)}{x} = \frac{0}{x} = 0 \underset{x\to 0}{\longrightarrow} 0,$$

hence $\partial_1 f(0,1) = 0$.

b) The following equivalent is well-known:

$$\sqrt{1+t}-1 \underset{t\to 0}{\sim} \frac{t}{2},$$

hence

$$\lim_{t\to 0} \frac{1-\sqrt{1+t}}{t} = -\frac{1}{2},$$

hence

$$\lim_{t\to 0} \frac{1-\sqrt{1+t}}{t} + 1 = \frac{1}{2}.$$

c) If f were differentiable at $(0,1)$, we would have $\mathrm{d}_{(0,1)}f = \ln(2)e_2'$ (where e_2' is the second vector of the standard dual basis of $\mathbb{R}^2$). But this can't be true, since, for $t \in \mathbb{R}^*$,

$$\frac{\left|f(0,1+t)-f(0,1)-\ln(2)t\right|}{\|(0,t)\|} = \left|\frac{\left(1+t-\sqrt{1+t}\right)\ln\left(t^2\right)}{t}\right| \underset{t\to 0}{\sim} \frac{1}{2}\left|\ln\left(t^2\right)\right| \underset{t\to 0}{\longrightarrow} +\infty.$$

We conclude that f is not differentiable at $(0,1)$.

Exercise 2.

1. We notice that ψ is linear, hence we only need to show that ψ_{f_0} is continuous at 0_E.

Let $h \in E$ and $x \in [-1,1]$. Then:

$$\left|\psi_{f_0}(h)(x)\right| = \left|\int_0^x f_0(t)h(t)\,\mathrm{d}t\right| \leq \left|\int_0^x \left|f_0(t)h(t)\right|\mathrm{d}t\right| \leq \left|\int_0^x \|f_0\|_\infty \|h\|_\infty \,\mathrm{d}t\right| \leq \|f_0\|_\infty \|h\|_\infty.$$

Hence

$$\left\|\psi_{f_0}(h)\right\|_\infty \leq \|f_0\|_\infty \|h\|_\infty \underset{\|h\|_\infty \to 0}{\longrightarrow} 0.$$

Hence ψ is continuous at 0_E, hence ψ is continuous.

2. Let $h \in E$. Then, for $x \in [-1,1]$,

$$\Phi(f_0+h)(x) = \int_0^x f_0(t)^2\,\mathrm{d}t + 2\int_0^x f_0(t)h(t)\,\mathrm{d}t + \int_0^x h(t)^2\,\mathrm{d}t = \Phi(f_0)(x) + 2\psi_{f_0}(h)(x) + \Phi(f_0)(h)(x),$$

hence

$$\Phi(f_0+h) = \Phi(f_0) + 2\psi_{f_0}(h) + \Phi(f_0)(h).$$

We already know that ψ_{f_0} is a linear continuous map, so in order to show that Φ is differentiable at f_0 we only need to show that:

$$\lim_{\|h\|_\infty \to 0} \frac{\left\|\Phi(h)\right\|_\infty}{\|h\|_\infty} = 0.$$

It is clear (from the computation we have already performed above), that

$$\left\|\Phi(h)\right\|_\infty \leq \|h\|_\infty^2$$

hence

$$\frac{\left\|\Phi(h)\right\|_\infty}{\|h\|_\infty} \leq \|h\|_\infty \underset{\|h\|_\infty \to 0}{\longrightarrow} 0.$$

Hence Φ is differentiable at f_0 and

$$D_{f_0}\Phi = 2\psi_{f_0}.$$

Exercise 3.

1. a) Let $(x, y) \in U$ and $(u, v) \in V$. Then:

$$\varphi(x,y) = (u,v) \iff \begin{cases} \dfrac{y}{x} = u \\ xy = v \end{cases} \iff \begin{cases} y^2 = uv \\ x^2 = \dfrac{v}{u} \end{cases} \iff \begin{cases} y = \sqrt{uv} \\ x = \sqrt{\dfrac{v}{u}} \end{cases} \quad \textit{(since } x \textit{ and } y \textit{ are positive).}$$

It is clear that the formulas we obtain for x and y will yield $(x, y) \in U$ whenever $(u, v) \in V$, hence φ is a bijection and

$$\begin{aligned} \varphi^{-1} : \quad V &\longrightarrow \quad U \\ (u,v) &\longmapsto \left(\sqrt{\frac{v}{u}}, \sqrt{uv}\right). \end{aligned}$$

Since φ is a bijection and φ is of class C^∞ and φ^{-1} is of class C^∞ (this is clear from the form of φ^{-1}), we conclude that φ is a C^∞-diffeomorphism.

b) i) Let $(x, y) \in U$. Then

$$J_{(x,y)}\varphi = \begin{pmatrix} -y/x^2 & 1/x \\ y & x \end{pmatrix}.$$

ii) We can use the Global Inverse Function Theorem: we know that φ is a bijection of class C^∞. Now,

$$\forall (x,y) \in U, \ \det J_{(x,y)}\varphi = -2\,\frac{y}{x} \neq 0$$

hence, by the Global Inverse Function Theorem, φ is a C^∞-diffeomorphism.

c) See Figure 2.

d) Let $A(1, 1/2)$.

i) $\varphi(A) = (1/2, 1/2)$.

ii) • The u-coordinate that passes through A is the curve of equation

$$\frac{y}{x} = \frac{1}{2}$$

i.e., of equation $y - x/2 = 0$. Hence a normal vector (of this level set) is

$$n_u = (-1/2, 1).$$

• The v-coordinate that passes through A is the curve of equation

$$xy = \frac{1}{2},$$

hence a normal vector (of this level set) is

$$n_v = (y, x)\,\Big|_{(x,y)=(1,1/2)} = (1/2, 1).$$

iii) See Figure **??**.

iv) We use the Jacobian matrix of φ at A, namely,

$$J_A\varphi = \begin{pmatrix} -1/2 & 1 \\ 1/2 & 1 \end{pmatrix},$$

to compute:

$$J_A\varphi \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 5 \\ 3 \end{pmatrix} \quad \text{and} \quad J_A\varphi \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3 \\ 5 \end{pmatrix},$$

hence

$$D_A\varphi(n_u) = \left(\frac{5}{4}, \frac{3}{4}\right) \quad \text{and} \quad D_A\varphi(n_v) = \left(\frac{3}{4}, \frac{5}{4}\right).$$

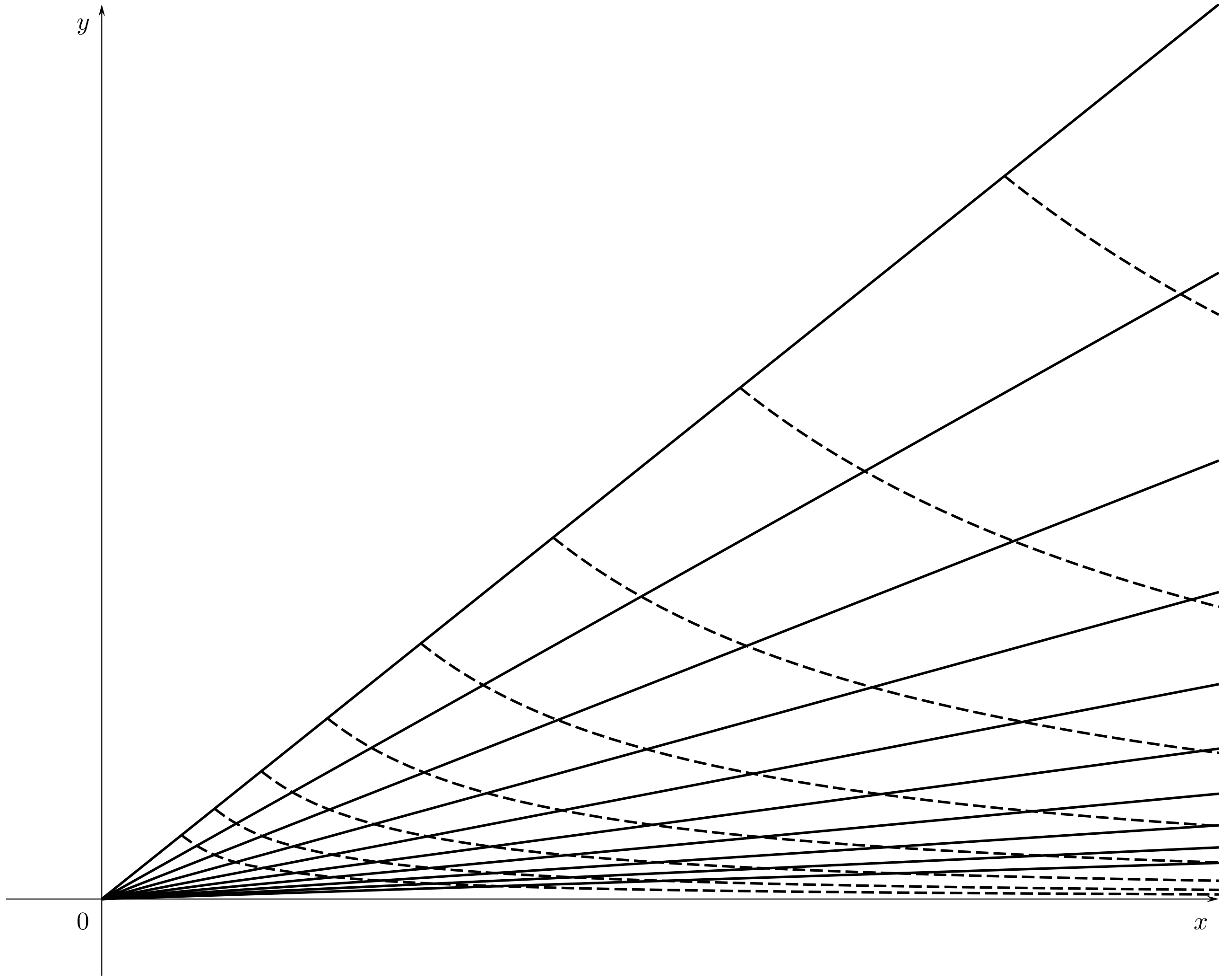

Figure 2. u-coordinates (plain) and v-coordinates (dashed)

e) Let $(u, v) \in V$.

i) We give the Jacobian of φ^{-1} at (u, v):

$$J_{(u,v)}\left(\varphi^{-1}\right) = \begin{pmatrix} -\dfrac{\sqrt{v}}{2u^{3/2}} & \dfrac{1}{2\sqrt{uv}} \\ \dfrac{\sqrt{v}}{2\sqrt{u}} & \dfrac{\sqrt{u}}{2\sqrt{v}} \end{pmatrix}.$$

ii) The relation is:

$$D_{(u,v)}\left(\varphi^{-1}\right) = \left(D_{\varphi^{-1}(u,v)}\varphi\right)^{-1}.$$

iii) We check it with the Jacobian matrices, i.e., we check that

$$J_{(u,v)}\left(\varphi^{-1}\right) = \left(J_{\varphi^{-1}(u,v)}\varphi\right)^{-1}.$$

We know that

$$J_{\varphi^{-1}(u,v)}\varphi = J_{(\sqrt{v/u},\sqrt{uv})}\varphi = \begin{pmatrix} -u^{3/2}/\sqrt{v} & \sqrt{u/v} \\ \sqrt{uv} & \sqrt{v/u} \end{pmatrix}.$$

Its determinant is

$$\det J_{\varphi^{-1}(u,v)}\varphi = -2u \neq 0.$$

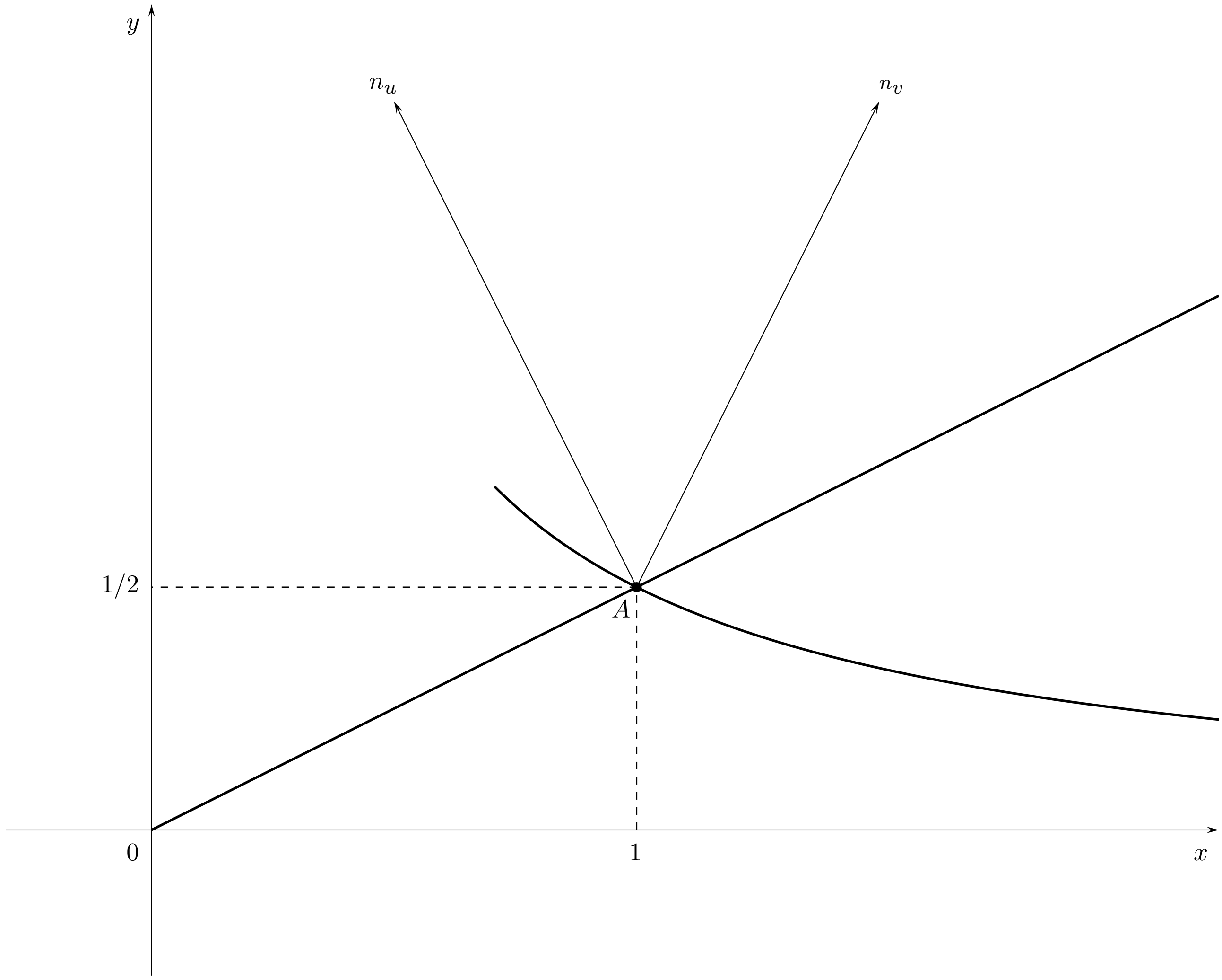

Figure 3. A, the u- and v-coordinates that pass through A and the vectors n_u and n_v

Hence

$$\left(J_{\varphi^{-1}(u,v)}\varphi\right)^{-1} = -\frac{1}{2u}\begin{pmatrix}\sqrt{v/u} & -\sqrt{u/v} \\ -\sqrt{uv} & -\frac{u^{3/2}}{\sqrt{v}}\end{pmatrix} = \begin{pmatrix}-\frac{\sqrt{v}}{2u^{3/2}} & \frac{1}{2\sqrt{uv}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}}\end{pmatrix}.$$

An we indeed obtain

$$\left(J_{\varphi^{-1}(u,v)}\varphi\right)^{-1} = J_{(u,v)}\left(\varphi^{-1}\right).$$

2. We have

$$\forall (x,y) \in U,\ f(x,y) = g\left(\frac{y}{x}, xy\right).$$

a)

$$\partial_1 f(x,y) = -\frac{y}{x^2}\partial_1 g\left(\frac{y}{x}, xy\right) + y\partial_2 g\left(\frac{y}{x}, xy\right),$$
$$\partial_2 f(x,y) = \frac{1}{x}\partial_1 g\left(\frac{y}{x}, xy\right) + x\partial_2 g\left(\frac{y}{x}, xy\right),$$

b)

$$x\partial_1 f(x,y) - y\partial_2 f(x,y) = -\frac{y}{x}\partial_1 g\left(\frac{y}{x}, xy\right) + xy\partial_2 g\left(\frac{y}{x}, xy\right) - \frac{y}{x}\partial_1 g\left(\frac{y}{x}, xy\right) - xy\partial_2 g\left(\frac{y}{x}, xy\right)$$

$$
\begin{aligned}
&= -\frac{2y}{x}\partial_1 g\left(\frac{y}{x}, xy\right) \\
&= -2u\partial_1 g(u,v)
\end{aligned}
$$

3. a) If f is a solution of $(*)$, then by the computation of the previous section, we have:

$$\forall (x,y) \in U,\ -2\frac{y}{x}\partial_1 g\left(\frac{y}{x}, xy\right) = xy$$

hence

$$\forall (x,y) \in U,\ \partial_1 g\left(\frac{y}{x}, xy\right) = -\frac{x^2}{2}$$

Now if $(u,v) \in V$, since φ is surjective, there exists $(x,y) \in U$ such that $\varphi(x,y) = (u,v)$. Hence we also have:

$$\forall (u,v) \in V,\ \partial_1 g(u,v) = -\frac{v}{2u},$$

i.e., g is a solution of $(*)$.

Conversely, if g is a solution of $(**)$ then: for $(x,y) \in U$,

$$x\partial_1 f(x,y) - y\partial_2 f(x,y) = -\frac{2y}{x}\partial_1 g\left(\frac{y}{x}, xy\right) = -\frac{2y}{x}\left(-\frac{x^2}{2}\right) = xy,$$

hence f is a solution of $(*)$.

b) If g is a solution of $(*)$, then there exists a function $A : (0,1) \to \mathbb{R}$ such that

$$\forall (u,v) \in V,\ g(u,v) = -\frac{v}{2}\ln(u) + A(v).$$

If we want g of class C^1, we just need to take A of class C^1.

c) By Question 3a) and since φ is a C^∞-diffeomorphism, f is a solution of class C^1 of $(*)$ if and only if there exists a function $A : (0,1) \to \mathbb{R}$ of class C^1 such that

$$\forall (x,y) \in U,\ f(x,y) = -\frac{xy}{2}\ln\left(\frac{y}{x}\right) + A(xy).$$