INSA | INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — S1 — Math Test #2 — 2h

December 10, 2018

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

The marks are given as a guide only; the final marking scheme might differ from the marks provided here.

Exercise 1 (*7 marks*). We define the set

$$D = \{(x,y) \in \mathbb{R}^2 \mid y > 0\},$$

$(\ln u)' = \frac{u'}{u}$

and the function

$$f : \begin{array}{rcl} D & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & \begin{cases} (y-\sqrt{y})\ln\left(x^2+(y-1)^2\right) & \text{if } (x,y) \in D \setminus \{(0,1)\} \\ 0 & \text{if } (x,y) = (0,1). \end{cases} \end{array}$$

1. Show that D is an open subset of $\mathbb{R}^2$.
2. a) Show that
$$\forall y \in \mathbb{R}_+^*,\ \left|y - \sqrt{y}\right| \le |y-1|.$$
   b) Deduce that
$$\forall (x,y) \in D,\ \left|y-\sqrt{y}\right| \le \left\|(x, y-1)\right\|_2.$$
   c) Show that f is continuous at $(0,1)$.
3. Let $(x,y) \in D \setminus \{(0,1)\}$.
   a) Compute the first order partial derivatives of f at (x,y).
   b) Is f of class C^1 on $D \setminus \{(0,1)\}$?
   c) Give the *first order* Taylor–Young expansion of f at $(2,1)$.
   d) Compute the directional derivative of f at $(2,1)$ in the direction $(1,1)$.
4. a) Compute $\partial_1 f(0,1)$.
   b) Prove that
$$\lim_{t\to 0} \frac{1+t-\sqrt{1+t}}{t} = \frac{1}{2}.$$
   c) Is the function f differentiable at $(0,1)$?

Exercise 2 (*3 marks*). Let $E = C([-1,1])$ be the vector space of continuous functions on $[-1,1]$ and let $\|\cdot\|_\infty$ be the ∞-norm on E. We define the mappings:

$$\Phi : \begin{array}{rcl} E & \longrightarrow & E \\ f & \longmapsto & \left(x \longmapsto \displaystyle\int_0^x f(t)^2\,\mathrm{d}t\right). \end{array}$$

and, for $f_0 \in E$,

$$\psi_{f_0} : \begin{array}{rcl} E & \longrightarrow & E \\ h & \longmapsto & \left(x \longmapsto \displaystyle\int_0^x f(t)h(t)\,\mathrm{d}t\right). \end{array}$$

You're given that Φ and ψ are well defined.
Let $f_0 \in E$.

1. Show that ψ_{f_0} is continuous.
2. Show that Φ is differentiable at f_0 and determine an explicit expression of $D_{f_0}\Phi$.

Exercise 3 (*10 marks*). We define the sets

$$U = \{(x, y) \in \mathbb{R}^2 \mid 0 < y < x\} \quad \text{and} \quad V = (0, 1) \times \mathbb{R}_+^*.$$

and the mapping

$$\begin{array}{rccc} \varphi : & U & \longrightarrow & V \\ & (x, y) & \longmapsto & \left(\dfrac{y}{x}, xy\right). \end{array}$$

You're given that U and V are open sets and that φ is well-defined, and of class C^∞.

1. a) Show that φ is a bijection and determine φ^{-1} explicitly. Deduce that φ is a C^∞-diffeomorphism.
   b) i) For $(x, y) \in U$, give the Jacobian matrix of φ.
      ii) Explain how to prove (again) that φ is a C^∞ diffeomorphism (without using the expression of φ^{-1} explicitly).
   c) Plot some coordinates associated with φ. More precisely, plot *in* U several curves of the form $y/x = u_0$ for several values of $u_0 \in (0, 1)$, that we shall call the u-coordinate, and of the form $xy = v_0$ for several values of $v_0 \in \mathbb{R}_+^*$, that we shall call the v-coordinate, in a different color for each form.
   d) Let $A = \cancel{(1/2, 1)}$. $A = (1, 1/2)$
      i) Compute the (u, v)-coordinates of A, that is $\varphi(A)$.
      ii) Give a non-nil normal vector n_u to the u-coordinate that passes through A and a non-nil normal vector n_v to the v-coordinate that passes through A.
      iii) On a figure (a separate one from that of Question 1c), plot A, the u- and v-coordinates that pass through A, and n_u and n_v.
      iv) Compute $D_A\varphi(n_u)$ and $D_A\varphi(n_v)$.
   e) Let $(u, v) \in V$.
      i) Compute *directly from the expression of* φ^{-1} *you obtained in Question 1(b)i* the expression of $D_{(u,v)}(\varphi^{-1})$.
      ii) What relation exists between $D_{(u,v)}(\varphi^{-1})$ and the differential of φ (at a point you will specify)?
      iii) Check that this relation is fulfilled.

2. Let $f : U \to \mathbb{R}$ be a function of class C^1. We set $g = f \circ \varphi^{-1}$. Obviously, $g : V \to \mathbb{R}$ is well-defined and of class C^1.
   a) Let $(x, y) \in U$. Compute the first order partial derivatives of f at (x, y) in terms of the first order partial derivatives of g at points you will specify.
   b) Let $(x, y) \in U$ and set $(u, v) = \varphi(x, y)$. Show that:

   $-2u\,\partial_1 g(u,v)$

   $$x\partial_1 f(x, y) - y\partial_2 f(x, y) = -\cancel{\frac{2}{u}}\partial_1 g(u, v).$$

3. In this question we study the general solution of class C^1 of the following partial differential equation on U:

   $$(*) \qquad \forall (x, y) \in U,\ x\partial_1 f(x, y) - y\partial_2 f(x, y) = xy.$$

   With no surprises, we will also study general solution of class C^1 of the following partial differential equation on V:

   $$(**) \qquad \forall (u, v) \in V,\ \partial_1 g(u, v) = \cancel{-\frac{2v}{u}}. \quad -\frac{v}{2u}$$

   a) Let f be a function of class C^1 and set $g = f \circ \varphi^{-1}$, like in the previous question. Show that f is a solution of $(*)$ if and only if g is a solution of $(**)$.
   b) Find the general solution g of $(**)$ of class C^1.
   c) Deduce the general solution f of $(*)$ of class C^1.