INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Solution of Math Test #3

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Exercise 1.

1. Let $(x,y)\in U$ and $(u,v)\in\mathbb{R}^2$. Then:

$$\varphi(x,y)=(u,v)\iff\begin{cases}xy=u\\2x=v\end{cases}\iff\begin{cases}y=u/x\qquad(x\neq 0)\\x=v/2\end{cases}\iff\begin{cases}y=2u/v\\x=v/2\end{cases}$$

Now,

$$v/2\in(0,1)\iff v\in(0,2)$$

and

$$2u/v>v/2\iff 4u>v^2.$$

Hence,

$$V=\bigl\{(u,v)\in\mathbb{R}^2\bigm| 0<v<2,\ u>v^2/2\bigr\}.$$

We now check that

$$\psi:\begin{array}{ccc}U&\longrightarrow&V\\(x,y)&\longmapsto&(xy,2x)\end{array}$$

is a C^2-diffeomorphism: from the previous computation, it is clear that ψ is a bijection, and that

$$\psi^{-1}:\begin{array}{ccc}V&\longrightarrow&U\\(u,v)&\longmapsto&(v/2,2u/v).\end{array}$$

It is clear that both ψ and ψ^{-1} are of class C^2 (even of class C^∞), hence ψ is a C^2-diffeomorphism.

See Figures 4 and 5.

2. Let $f:U\to\mathbb{R}$ be a function of class C^2 and define $g=f\circ\psi^{-1}$. Since ψ^{-1} is of class C^2, by composition, g is also of class C^2. Now, $f=g\circ\psi$, and we have, for $(x,y)\in U$,

$$\begin{aligned}f(x,y)&=g(xy,2x),\\\partial_1f(x,y)&=y\partial_1g(xy,2x)+2\partial_2g(xy,2x),\\\partial_2f(x,y)&=x\partial_1g(xy,2x),\\\partial^2_{1,2}f(x,y)&=\partial_1g(xy,2x)+xy\partial^2_{1,1}g(xy,2x)+2x\partial^2_{1,2}g(xy,2x)\\\partial^2_{2,2}f(x,y)&=x^2\partial^2_{1,1}g(xy,2x),\end{aligned}$$

and hence

$$\begin{aligned}x\partial^2_{1,2}f(x,y)-y\partial^2_{2,2}f(x,y)-\partial_2f(x,y)&=x\bigl(\partial_1g(xy,2x)+xy\partial^2_{1,1}g(xy,2x)+2x\partial^2_{1,2}g(xy,2x)\bigr)\\&\quad-x^2y\partial^2_{1,1}g(xy,2x)\\&\quad-x\partial_1g(xy,2x)\\&=2x^2\partial^2_{1,2}g(xy,2x).\end{aligned}$$

Hence,

$$\begin{aligned}f\text{ is a solution of (E)}&\iff\forall(x,y)\in U,\ 2x^2\partial^2_{1,2}g(xy,2x)=2x^3y\\&\iff\forall(x,y)\in U,\ \partial^2_{1,2}g(xy,2x)=xy\\&\iff\forall(u,v)\in V,\ \partial^2_{1,2}g(u,v)=u\\&\iff\exists a:(0,2)\to\mathbb{R},\ \forall(u,v)\in V,\ \partial_2g(u,v)=\frac{u^2}{2}+a(v)\\&\iff\exists A:(0,2)\to\mathbb{R},\ \exists B:\mathbb{R}_+^*\to\mathbb{R},\ \forall(u,v)\in V,\ g(u,v)=\frac{u^2v}{2}+A(v)+B(u)\\&\iff\exists A:(0,2)\to\mathbb{R},\ \exists B:\mathbb{R}_+^*\to\mathbb{R},\ \forall(x,y)\in U,\ f(x,y)=x^3y^2+A(2x)+B(xy).\end{aligned}$$

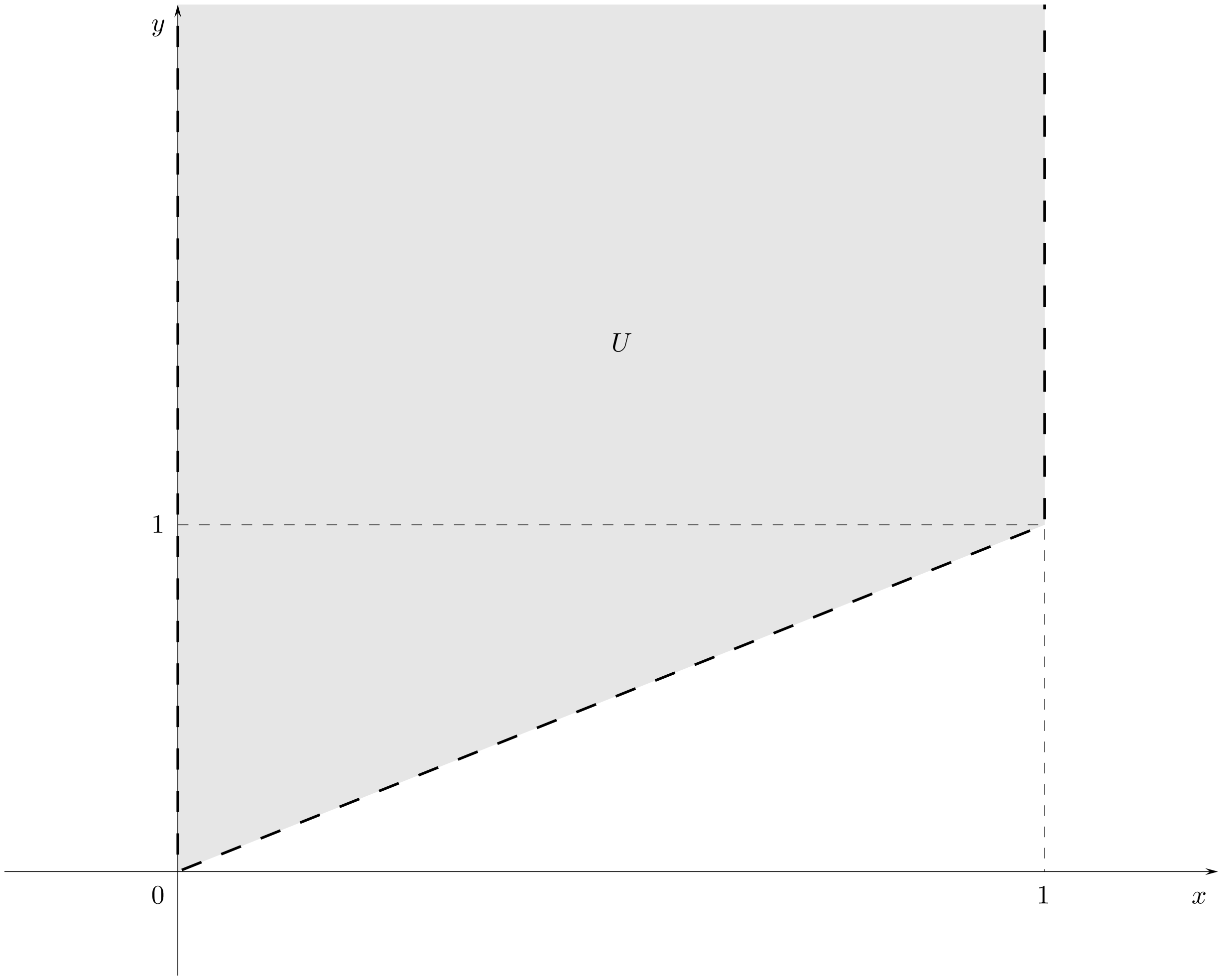

Figure 4. Open set U of Exercise 1.

Hence the general solution of (E) is:

$$f(x,y) = x^3y^2 + A(2x) + B(xy),$$

where $A : (0,2) \to \mathbb{R}$ and $B : \mathbb{R}_+^* \to \mathbb{R}$ are functions of class C^2.

Exercise 2.

1. We apply the Implicit Function Theorem to f at O:

   - The function f is of class C^2 (or even of class C^∞),
   - $f(0,0) = 0$,
   - $\partial_2 f(0,0) = 2 \neq 0$,

   hence, by the Implicit Function Theorem, there exists an open interval U containing 0 and an open interval V containing 0 and a mapping $\varphi : U \to V$ of class C^2 such that

   $$\forall (x,y) \in U \times V,\ \Big(f(x,y) = 0 \iff y = \varphi(x)\Big).$$

   We also have $\varphi(0) = 0$.

2. a) For $(x,y) \in \mathbb{R}^2$,

   $$\partial_1 f(x,y) = 4x - 2 \qquad \text{and} \qquad \partial_2 f(x,y) = -2\big(2y^3 - y + 1\big)\big(6y^2 - 1\big),$$

   hence,

   $$\forall x \in U,\ \varphi'(x) = \frac{2x-1}{\big(2\varphi(x)^3 - \varphi(x) + 1\big)\big(6\varphi(x)^2 - 1\big)}$$

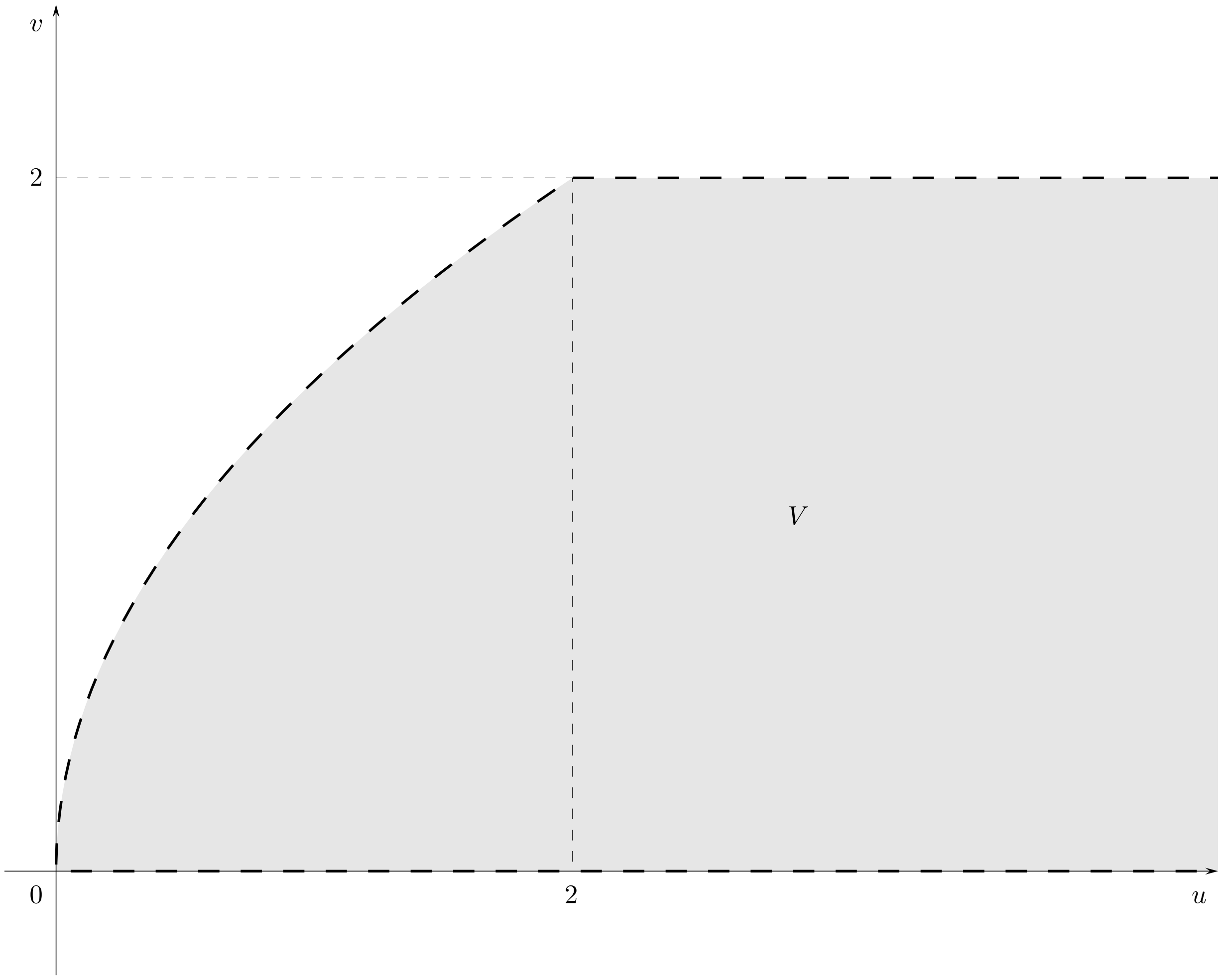

Figure 5. Open set V of Exercise 1.

In particular, since $\varphi(0) = 0$,

$$\varphi'(0) = 1.$$

There are lots of possibilities to get the value of $\varphi''(0)$. For example, we define

$$\forall x \in U,\ D(x) = \big(2\varphi(x)^3 - \varphi(x) + 1\big)\big(6\varphi(x)^2 - 1\big).$$

Then, $D(0) = -1$ and $D'(0) = \varphi'(0) = 1$, hence, by the quotient rule,

$$\varphi''(0) = \frac{2D(0) + D'(0)}{D(0)^2} = -1.$$

We conclude that:

$$\varphi(x) \underset{x\to 0}{=} x - \frac{x^2}{2} + o\big(x^2\big).$$

b) Hence, an equation of the tangent line to $\mathscr{C}$ at O is:

$$\Delta\colon y = x,$$

and, in a neighborhood of O, the curve $\mathscr{C}$ lies below Δ.

3. a) Let $A(x, y) \in \mathbb{R}^2$. The symmetric of A with respect to the straight line $x = 1/2$ is $A'(1 - x, y)$. Now,

$$\begin{aligned} f(A') = f(1-x, y) &= 2(1-x)^2 - 2(1-x) + 1 - \big(1 - y + 2y^3\big)^2 \\ &= 2 - 4x + 2x^2 - 2 + 2x + 1 - \big(1 - y + 2y^3\big)^2 = 2x^2 - 2x + 1 - \big(1 - y + 2y^3\big)^2 \\ &= f(A), \end{aligned}$$

hence:

$$A \in \mathscr{C} \iff f(A) = 0 \iff f(A') = 0 \iff A' \in \mathscr{C}.$$

b) See Figure 6.

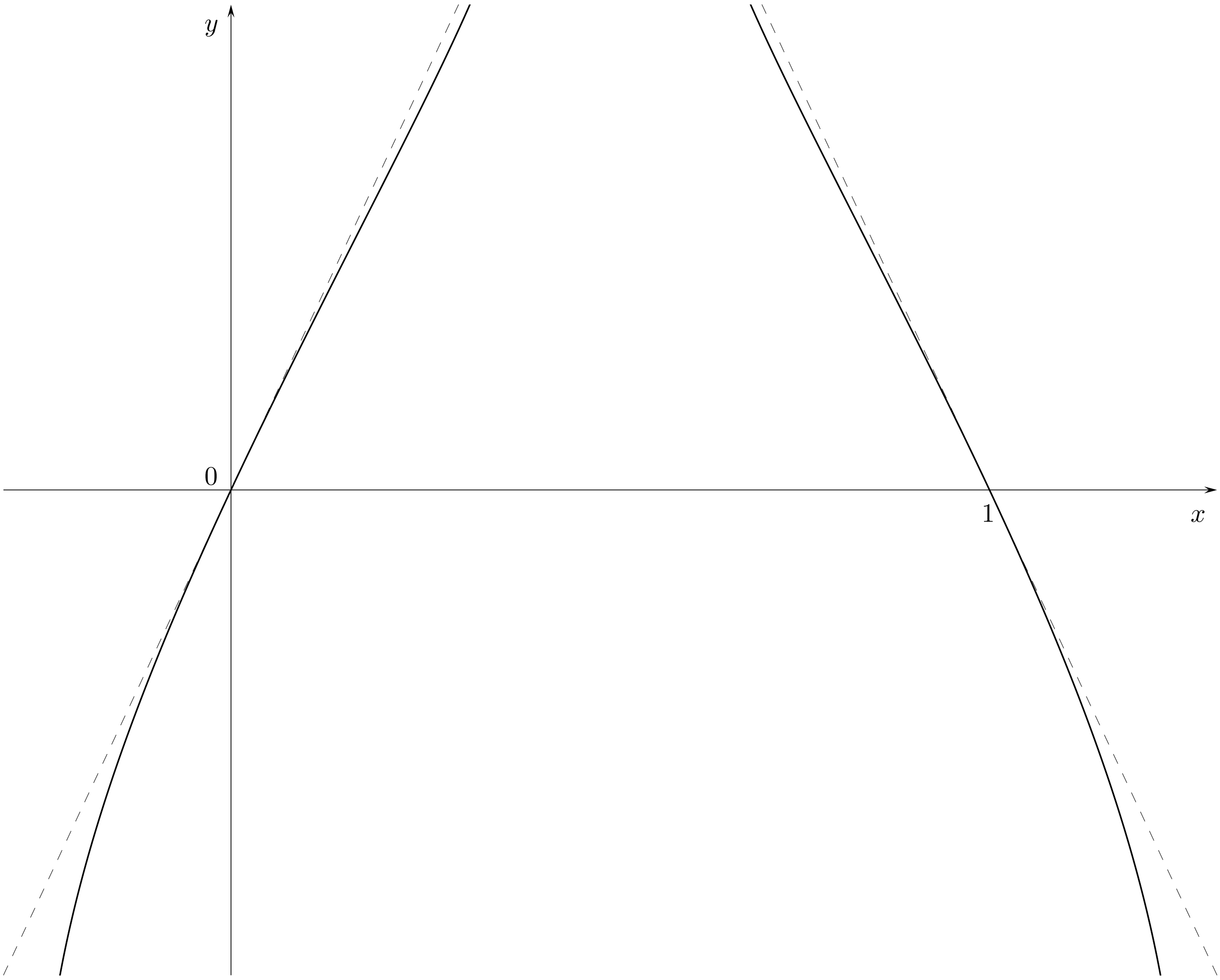

Figure 6. Curve $\mathscr{C}$ of Exercise 2 around O and $A(1,0)$, and the tangent lines (dashed) at O and A.

c) By symmetry, the equation of the tangent line to $\mathscr{C}$ at $A(1,0)$ is:

$$y = -x + 1.$$

Note. We have included a zoomed out version of $\mathscr{C}$ in Figure 7.

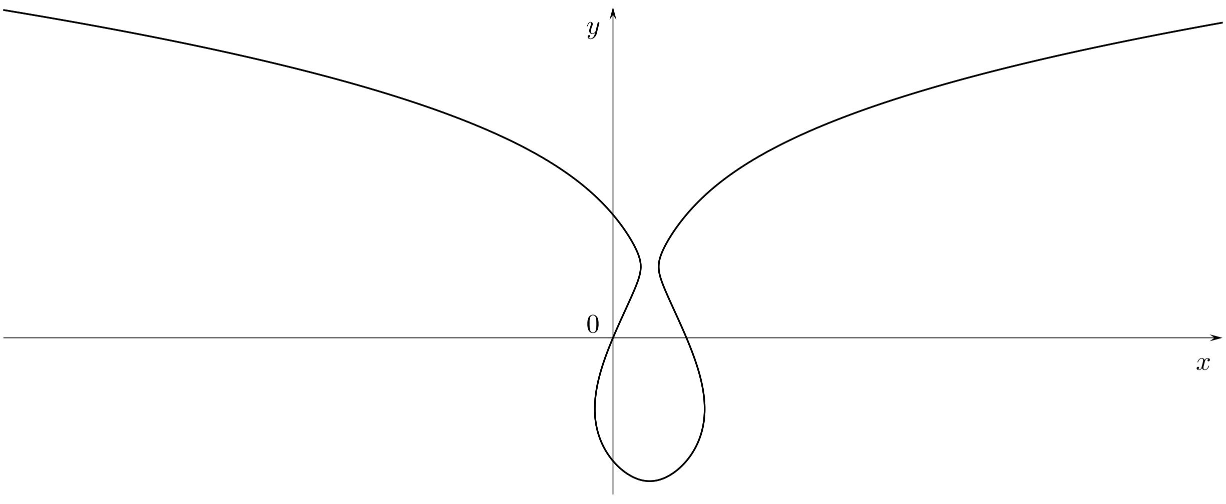

Figure 7. Curve $\mathscr{C}$ of Exercise 2.

Exercise 3.

1. We apply the Implicit Function Theorem:

   - The function f is clearly of class C^2 (and even of class C^∞),
   - $f(0,0,0)=0$,
   - $\partial_2 f(0,0,0)=1\neq 0$,

   hence, by the Implicit Function Theorem, there exists an open subset U of $\mathbb{R}^2$ containing $(0,0)$ and an open interval V containing 0 and a function $\varphi : U\to V$ of class C^2 such that $\varphi(0,0)=0$ and

$$\forall (x,z)\in U,\ \forall y\in V,\ \bigl(f(x,y,z)=0 \iff y=\varphi(x,z)\bigr).$$

2. a) $\overrightarrow{\operatorname{grad}} f(0,0,0)=(1,1,0)$, hence an equation of $\mathscr{P}$ is:

$$\mathscr{P}\colon x+y=0.$$

   b)

$$\partial_1\varphi(0,0)=-\frac{\partial_1 f\bigl(0,\varphi(0,0),0\bigr)}{\partial_2 f\bigl(0,\varphi(0,0),0\bigr)}=-1$$

   and

$$\partial_2\varphi(0,0)=-\frac{\partial_3 f\bigl(0,\varphi(0,0),0\bigr)}{\partial_2 f\bigl(0,\varphi(0,0),0\bigr)}=0,$$

   hence the first order Taylor–Young expansion of φ at $(0,0)$ is:

$$\varphi(x,z)\underset{(x,z)\to(0,0)}{=}-x+o\bigl(\|(x,y)\|\bigr)$$

   hence an equation of the tangent plane to $\mathscr{C}$ at $(0,0)$ is:

$$\mathscr{P}: y=-x.$$

3. From the "moreover" part of the Implicit Function Theorem, we know that, for all $(x,z)\in U$,

$$\partial_1\varphi(x,z)=-\frac{\partial_1 f\bigl(x,\varphi(x,z),z\bigr)}{\partial_2 f\bigl(x,\varphi(x,z),z\bigr)}=-\frac{z\mathrm{e}^{\varphi(x,z)}+\mathrm{e}^z}{xz\mathrm{e}^{\varphi(x,z)}+\mathrm{e}^z}$$

   In particular,

$$\partial_1\varphi(0,z)=-\frac{z\mathrm{e}^{\varphi(0,z)}+\mathrm{e}^z}{\mathrm{e}^z}=-z\mathrm{e}^{\varphi(0,z)-z}-1$$

   hence

$$\partial^2_{2,1}\varphi(0,0)=\lim_{z\to 0}\frac{\varphi(0,z)-\varphi(0,0)}{z}=\lim_{z\to 0}-\mathrm{e}^{\varphi(0,z)-z}=-1.$$

   Since φ is of class C^2, $\partial^2_{1,2}\varphi(0,0)=\partial^2_{2,1}\varphi(0,0)=-1$.

Exercise 4.

1. Since $1/n\underset{n\to+\infty}{\longrightarrow}0$,

$$\frac{\mathrm{e}^{1/n}-1}{n^\alpha}\underset{n\to+\infty}{\sim}\frac{1/n}{n^\alpha}=\frac{1}{n^{\alpha+1}}>0.$$

   Now, by Riemann, the series $\sum_n 1/n^{\alpha+1}$ converges if and only $\alpha+1>1$, i.e., $\alpha>0$. Hence, by the equivalent test, the series converges if and only if $\alpha>0$.

2. Define

$$\forall n\in\mathbb{N}^*,\ u_n=\frac{1}{\sqrt{n}+2n}>0.$$

   The series we're studying is $\sum_n(-1)^n u_n$, which is an alternating series. Now,

   - $\lim\limits_{n\to+\infty} u_n=0$,

- the sequence $(u_n)_{n\in\mathbb{N}^*}$ is decreasing,

hence, by the alternating series test, the series $\sum_n (-1)^n u_n$ converges.

Moreover, by the alternating series test, we know that the sign of R_N is that of u_{N+1}, that is:

- If N is odd, then $R_N \geq 0$,
- If N is even, then $R_N \leq 0$.

Moreover, we have:

$$|R_N| \leq |u_{N+1}| = \frac{1}{\sqrt{N+1} + 2(N+1)}.$$

3. a)

$$\frac{1}{1+n^2} \underset{n\to+\infty}{\sim} \frac{1}{n^2} > 0.$$

Now by Riemann, we know that the series $\sum_n 1/n^2$ converges hence, by the equivalent test, the series we're studying is convergent.

b) Define the function

$$f : \begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \dfrac{1}{1+t^2}. \end{array}$$

- the function f is non-increasing,
- $$\forall n \in \mathbb{N},\ \frac{1}{1+n^2} = f(n),$$

hence, by the Integral Comparison Test, for $N \in \mathbb{N}^*$,

$$\int_{N+1}^{+\infty} f(t)\,\mathrm{d}t \leq S - S_N = \sum_{n=N+1}^{+\infty} f(n) \leq \int_N^{+\infty} f(t)\,\mathrm{d}t.$$

Now, for $a, b \in \mathbb{R}_+^*$,

$$\int_a^b \frac{\mathrm{d}t}{1+t^2} = \arctan(b) - \arctan(a),$$

hence

$$\int_a^{+\infty} \frac{\mathrm{d}t}{1+t^2} = \frac{\pi}{2} - \arctan(a) = \arctan\left(\frac{1}{a}\right).$$

Hence, for $N \in \mathbb{N}^*$,

$$\arctan\left(\frac{1}{N+1}\right) \leq S - S_N = \sum_{n=N+1}^{+\infty} f(n) \leq \arctan\left(\frac{1}{N}\right).$$