INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — S1 — Math Test #3 — 2h

January 29, 2019

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

The marks are given as a guide only; the final marking scheme might differ from the marks provided here.

Exercise 1 is common with PCC2.

$5 \times (20+4)$
$= 100 + 20 = 120$

Exercise 1 (*7 marks*). Let

$$\begin{array}{rccc} \varphi: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (x,y) & \longmapsto & (xy, 2x). \end{array}$$

1. Define the open set

$$U = \{(x,y) \in \mathbb{R}^2 \,|\, 0 < x < 1,\ y > x\}.$$

   Find an open subset V of $\mathbb{R}^2$ such that the restriction of φ to U is a C^2-diffeomorphism from U to V. Represent U and V on two distinct figures.

2. Determine the functions $f : U \longrightarrow \mathbb{R}$ of class C^2 that are solution of the following partial differential equation:

$$x\frac{\partial^2 f}{\partial x \partial y}(x,y) - y\frac{\partial^2 f}{\partial y^2}(x,y) - \frac{\partial f}{\partial y}(x,y) = 2x^3 y.$$

$$x\,\partial_{1,2}^2 f(x,y) - y\,\partial_{2,2}^2 f(x,y) - \partial_2 f(x,y) = 2x^3 y$$

Exercise 2 (*5 marks*). Let

$$\begin{array}{rccc} f: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & 2x^2 - 2x + 1 - (1 - y + 2y^3)^2. \end{array}$$

We denote by $\mathscr{C}$ the curve of $\mathbb{R}^2$ of equation

$$(\mathscr{C}) \qquad f(x,y) = 0.$$

1. Show that in a neighborhood of $O(0,0)$, the curve $\mathscr{C}$ admits a representation of the form $y = \varphi(x)$ with φ of class C^2.
2. a) Determine the second-order Taylor–Young expansion of φ at 0.
   b) Deduce an equation of the tangent line Δ to $\mathscr{C}$ at O and the relative position of $\mathscr{C}$ with respect to Δ.
3. a) Show that the curve $\mathscr{C}$ is symmetric with respect to the straight line of equation $x = 1/2$.
   b) Sketch (on the same figure) the curve $\mathscr{C}$ in a neighborhood of O and of $A(1,0)$.
   c) Give (without any justifications) the equation of the tangent line to $\mathscr{C}$ at $A(1,0)$.

0

Exercise 3 (*4 marks*). Let

$$\begin{array}{rccl} f: & \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ & (x, y, z) & \longmapsto & xze^y + (x+y)e^z. \end{array}$$

We denote by $\mathscr{S}$ the surface of $\mathbb{R}^3$ of equation

$$(\mathscr{S}) \qquad f(x, y, z) = 0.$$

1. Show that in a neighborhood of $O(0, 0, 0)$, the surface $\mathscr{S}$ admits a representation of the form $y = \varphi(x, z)$ with φ of class C^2.
2. Determine an equation of the tangent plane $\mathscr{P}$ to $\mathscr{S}$ at O, using two different methods:
   a) using the gradient of f at O,
   b) using the first order Taylor–Young expansion of φ at $(0, 0)$.
3. Determine the value of $\partial^2_{1,2}\varphi(0, 0)$.

Exercise 4 (*4 marks*). The questions of this exercise are independent from each other

1. Let $\alpha \in \mathbb{R}$. Determine the convergence of the following series:

$$\sum_n \frac{e^{1/n} - 1}{n^\alpha}.$$

2. Show that the following series is convergent:

$$\sum_n (-1)^n \frac{1}{\sqrt{n} + 2n}.$$

   For $N \in \mathbb{N}^*$, determine the sign of

$$R_N = \sum_{n=N+1}^{+\infty} (-1)^n \frac{1}{\sqrt{n} + 2n},$$

   as well as an upper bound of $|R_N|$.

3. a) Briefly justify that the following series is convergent:

$$\sum_n \frac{1}{1 + n^2}.$$

   b) For $N \in \mathbb{N}$ we define

$$S_N = \sum_{n=0}^{N} \frac{1}{1 + n^2}.$$

   We also define:

$$S = \sum_{n=0}^{+\infty} \frac{1}{1 + n^2}.$$

   Show, using the integral comparison test that

$$\forall N \in \mathbb{N},\ \arctan\left(\frac{1}{N+1}\right) \leq S - S_N \leq \arctan\left(\frac{1}{N}\right).$$

$$\frac{\dfrac{e^{\frac{1}{n+1}} - 1}{(n+1)^\alpha}}{\dfrac{e^{1/n} - 1}{n^\alpha}} = \frac{e^{\frac{1}{n+1}} - 1}{e^{1/n} - 1}$$