INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Solution of Math Test #4

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Exercise 1.

1. We recognize a geometric series of ratio $\mathrm{e}z^2$, hence $R = \mathrm{e}^{-1/2}$.
2. We know that the radius of convergence $R = 3$ of the power series $\sum_n (a_n + b_n)z^n$ satisfies:

$$R = 3 \geq \min\{R_a, R_b\},$$

and that if $R_a = 1 \neq R_b$ then $R = 3 = \min\{R_a, R_b\}$. This last statement being wrong since $\min\{R_a, R_b\} \leq 1$, we conclude that $R_b = 1$.

3. For $x \in (-2, +\infty)$ we have:

$$x \ln(2 + x) = x \left(\ln\left(1 + \frac{x}{2}\right) + \ln(2)\right)$$

and we conclude that for all $x \in (-2, 2]$,

$$\begin{aligned} x \ln(2 + x) &= x \left(\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} \left(\frac{x}{2}\right)^n \ + \ \ln(2)\right) \\ &= \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{2^n n} x^{n+1} \ + \ x\ln(2) \\ &= x\ln(2) + \sum_{n=2}^{+\infty} \frac{(-1)^n}{2^{n-1}(n-1)} x^n. \end{aligned}$$

4. a) The radius of convergence of this power series is 1 (magic lemma) and, for $x \in (-1, 1) \setminus \{0\}$,

$$\begin{aligned} \sum_{n=0}^{+\infty} \frac{x^n}{n+1} &= \sum_{n=1}^{+\infty} \frac{x^{n-1}}{n} \\ &= \frac{1}{x} \sum_{n=1}^{+\infty} \frac{x^n}{n} \\ &= -\frac{1}{x} \ln(1 - x) \end{aligned}$$

and if $x = 0$, the value of the sum is 1. Hence

$$\forall x \in (-1, 1),\ \sum_{n=0}^{+\infty} \frac{x^n}{n+1} = \begin{cases} -\dfrac{\ln(1-x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

b) The radius of convergence of this power series is $R = +\infty$. For $x \in \mathbb{R}$,

$$\begin{aligned} \sum_{n=1}^{+\infty} \frac{x^{2n}}{(n-1)!} &= \sum_{n=0}^{+\infty} \frac{x^{2(n+1)}}{n!} \\ &= x^2 \sum_{n=0}^{+\infty} \frac{x^{2n}}{n!} \\ &= x^2 \sum_{n=0}^{+\infty} \frac{\left(x^2\right)^n}{n!} \\ &= x^2 \mathrm{e}^{x^2}. \end{aligned}$$

(This equality is also valid for $x \in \mathbb{C}$).

Exercise 2.

1. Let $x \in (-1,1)$. Then

$$\frac{1}{(1-x)^2} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{+\infty} x^n\right) = \sum_{n=1}^{+\infty} n x^{n-1},$$

hence

$$\frac{x^3}{(1-x)^2} = \sum_{n=1}^{+\infty} n x^{n+2} = \sum_{n=3}^{+\infty} (n-2)x^n.$$

2. a) Let y be a function defined by a power series of radius $R > 0$, say

$$y : (-R,R) \longrightarrow \mathbb{R}, \quad x \longmapsto \sum_{n=0}^{+\infty} a_n x^n.$$

Then, for $x \in (-R,R)$,

$$x^2 y''(x) = \sum_{n=0}^{+\infty} n(n-1)a_n x^n$$

hence

$$x^2 y''(x) - 2y(x) = \sum_{n=0}^{+\infty} (n^2 - n - 2)a_n x^n = \sum_{n=0}^{+\infty} (n-2)(n+1)a_n x^n.$$

Hence, by the identity theorem,

$$y \text{ is a solution of (E)} \iff a_0 = a_1 = 0,\ \forall n \geq 3,\ (n-2)(n+1)a_n = n-2 \iff a_0 = a_1 = 0,\ \forall n \geq 3,\ a_n = \frac{1}{n+1}.$$

(There are no constraints on a_2).
By the magic lemma, we conclude that such solutions will have a radius of convergence of 1.

b) All our solutions satisfy $a_0 = a_1 = 0$, hence the graphs of the solutions pass through $(0,0)$ and have a horizontal tangent line there. The relative position of the graphs of the solutions with respect to this horizontal tangent line depends on a_2:

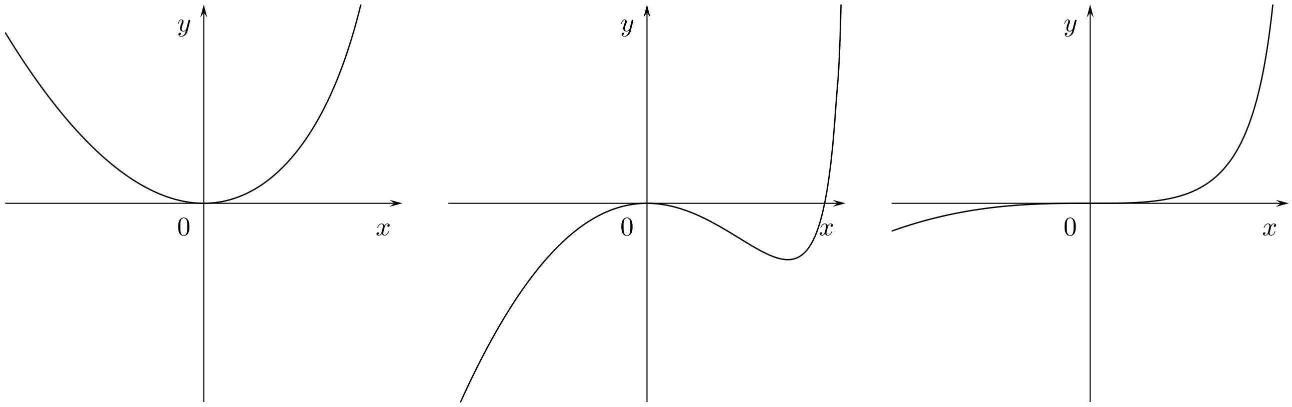
- if $a_2 > 0$: the graph of the corresponding solution lies (in a neighborhood of 0) above the tangent line at $(0,0)$; see Figure 8a,
- if $a_2 < 0$: the graph of the corresponding solution lies (in a neighborhood of 0) below the tangent line at $(0,0)$; see Figure 8b,
- if $a_2 = 0$: in this case we have to look at the next non-nil coefficient, which happens to be $a_3 = 1/4$: hence the graph of the corresponding solution crosses the tangent line at $(0,0)$: in a neighborhood of 0, the graph is below the tangent line to the left of 0 and above the tangent line to right left of 0; see Figure 8c.

c) The solutions y of (E) that possess a power series expansion have a radius of convergence of 1 and have the form:

$$y(x) = -1 - \frac{x}{2} + cx^2 + \varphi(x),$$

where $c \in \mathbb{R}$ and φ is the function obtained in Question 4a of Exercise 1:

$$\forall x \in (-1,1),\ \varphi(x) = \begin{cases} -\dfrac{\ln(1-x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$



(a) case $a_2 > 0$ (shown is $a_2 = 1$) **(b)** case $a_2 < 0$ (shown is $a_2 = -1$) **(c)** case $a_2 = 0$

Figure 8. Form of the solutions of Equation (E) that possess a power series expansion, in the cases $a_2 > 0$, $a_2 < 0$ and $a_2 = 0$.

Exercise 3.

1. Let $\alpha \in \mathbb{R}_+^* \setminus \{1/\mathrm{e}\}$. We denote by $(u_n)_{n\in\mathbb{N}}$ the general term of the series:

$$\forall n \in \mathbb{N},\ u_n = \frac{(n\alpha)^n}{n!}.$$

Since the series $\sum_n u_n$ has positive terms, we can use the ratio test: for $n \in \mathbb{N}$,

$$\frac{u_{n+1}}{u_n} = \frac{\big((n+1)\alpha\big)^{n+1}}{(n+1)!}\frac{n!}{(n\alpha)^n} = \left(\frac{n+1}{n}\right)^n \alpha \underset{n\to+\infty}{\longrightarrow} \mathrm{e}\alpha.$$

Hence:

- if $\alpha > 1/\mathrm{e}$, then the series diverges,
- if $\alpha < 1/\mathrm{e}$, then the series converges.

2. a) From the computation of Question 1,

$$\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = 1,$$

hence we're in the case where the ratio test fails.

b) Let $n \in \mathbb{N}$. Then:

$$\frac{u_{n+1}}{u_n} = \left(\frac{n+1}{n}\right)^n \mathrm{e}^{-1} = \exp\left(n\ln\left(1+\frac{1}{n}\right) - 1\right).$$

Now,

$$\ln\left(1+\frac{1}{n}\right) \underset{n\to+\infty}{=} \frac{1}{n} - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right),$$

hence

$$n\ln\left(1+\frac{1}{n}\right) - 1 \underset{n\to+\infty}{=} -\frac{1}{2n} + o\left(\frac{1}{n}\right),$$

hence

$$\frac{u_{n+1}}{u_n} \underset{n\to+\infty}{=} 1 - \frac{1}{2n} + o\left(\frac{1}{n}\right).$$

c) i) For $n \in \mathbb{N}^*$,

$$\begin{aligned} v_{n+1} - v_n &= \ln\big((n+1)u_{n+1}\big) - \ln(nu_n) \\ &= \ln\left(\frac{n+1}{n}\frac{u_{n+1}}{u_n}\right) \\ &\underset{n\to+\infty}{\sim} \frac{n+1}{n}\frac{u_{n+1}}{u_n} - 1 && \textit{since } \frac{n+1}{n}\frac{u_{n+1}}{u_n} \underset{n\to+\infty}{\longrightarrow} 1 \end{aligned}$$

Now

$$\begin{aligned}\frac{n+1}{n}\frac{u_{n+1}}{u_n}-1 &\underset{n\to+\infty}{=} \left(1+\frac{1}{n}\right)\left(1-\frac{1}{2n}+o\left(\frac{1}{n}\right)\right)-1\\ &\underset{n\to+\infty}{=} 1+\frac{1}{2n}+o\left(\frac{1}{n}\right)-1=\frac{1}{2n}+o\left(\frac{1}{n}\right)\\ &\underset{n\to+\infty}{\sim} \frac{1}{2n}.\end{aligned}$$

Hence

$$v_{n+1}-v_n \underset{n\to+\infty}{\sim} \frac{1}{2n}.$$

ii) By the equivalent test, since $1/2n>0$ is the general term of a divergent series, we conclude that the series $\sum_n(v_{n+1}-v_n)$ diverges.

iii) The series $\sum_n(v_{n+1}-v_n)$ is a series with positive terms (at least from a certain index), as shown by the equivalent obtained in Question 2c)i), hence the sequence of the partial sums is increasing; this sequence being divergent, its limit must be $+\infty$:

$$\lim_{N\to+\infty}\sum_{n=1}^{N}(v_{n+1}-v_n)=+\infty.$$

Now, for $N\in\mathbb{N}^*$,

$$\sum_{n=1}^{N}(v_{n+1}-v_n)=v_{N+1}-v_1,$$

and we conclude that $\lim\limits_{N\to+\infty} v_N=+\infty$.

d) Since $v_n \underset{n\to+\infty}{\longrightarrow} +\infty$, we conclude that $nu_n=\mathrm{e}^{v_n}\underset{n\to+\infty}{\longrightarrow}+\infty$. Hence there exists $N\in\mathbb{N}$ such that

$$\forall n\geq N,\ nu_n\geq 1,$$

i.e.,

$$\forall n\geq N,\ u_n\geq\frac{1}{n}>0,$$

and we conclude, by the comparison test, that the series $\sum_n u_n$ diverges.

Exercise 4.

1.

$$A=\begin{pmatrix}0&1&0\\1&1&-2\\0&-2&2\end{pmatrix}.$$

2.

$$\begin{array}{rccc}\varphi: & \mathbb{R}^3\times\mathbb{R}^3 & \longrightarrow & \mathbb{R}\\ & \big((x,y,z),(x',y',z')\big) & \longmapsto & xy'+x'y+yy'-2(yz'+y'z)+2zz'.\end{array}$$

3. We need to evaluate φ at the vectors of $\mathscr{C}$:

$$\varphi(u_1,u_1)=1,\quad \varphi(u_1,u_2)=0,\quad \varphi(u_1,u_3)=0,\quad \varphi(u_2,u_2)=-1,\quad \varphi(u_2,u_3)=0,\quad \varphi(u_3,u_3)=2.$$

(Note that since φ is symmetric, we only have 6 terms to compute). Hence

$$A'=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&2\end{pmatrix}.$$

4. The change of basis matrix is:

$$[\mathscr{C}]_{\mathrm{std}}=\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}.$$

The change of basis formula is:

$$A' = [\varphi]_{\mathscr{C}} = {}^t[\mathscr{C}]_{\text{std}} A [\mathscr{C}]_{\text{std}},$$

and we obtain:

$$A' = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & -2 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(insert happy smiley here).

5. Let $(x, y, z) \in \mathbb{R}^3$. Then:

$$(x, y, z) \in \{u_1 + u_2\}^{\perp} \iff \varphi\big((x, y, z), u_1 + u_2\big) = 0 \iff \varphi\big((x, y, z), (1, 2, 2)\big) = 0 \iff 2x - y = 0.$$

Hence

$$F^{\perp} = \big\{(x, y, z) \in \mathbb{R}^3 \;\big|\; 2x - y = 0\big\}.$$

Exercise 5.

1. Let $u, v, w \in E$ and let $\lambda \in \mathbb{R}$. Then:

$$\begin{aligned} \psi(u + \lambda v, w) &= \varphi\big(u + \lambda v, f(w)\big) \\ &= \varphi\big(u, f(w)\big) + \lambda\varphi\big(v, f(w)\big) && \textit{since } \varphi \textit{ is bilinear} \\ &= \psi(u, w) + \lambda\psi(v, w), \end{aligned}$$

hence ψ is linear with respect to its first argument, and

$$\begin{aligned} \psi(u, v + \lambda w) &= \varphi\big(u, f(v + \lambda w)\big) \\ &= \varphi\big(u, f(v) + \lambda f(w)\big) && \textit{since } f \textit{ is linear} \\ &= \varphi\big(u, f(v)\big) + \lambda\varphi\big(u, f(w)\big) && \textit{since } \varphi \textit{ is bilinear} \\ &= \psi(u, v) + \lambda\psi(u, w), \end{aligned}$$

hence ψ is linear with respect to its second argument.

Hence ψ is a bilinear form on E.

2. For $u, v \in E$,

$${}^t[u]_{\mathscr{B}} B [v]_{\mathscr{B}} = \psi(u, v) = \varphi\big(u, f(v)\big) = {}^t[u]_{\mathscr{B}} A \big[f(v)\big]_{\mathscr{B}} = {}^t[u]_{\mathscr{B}} AM \big[v\big]_{\mathscr{B}}.$$

We conclude (by uniqueness of the matrix of a bilinear form) that $B = AM$.