INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — S2 — Math Test #4 — 2h

April 1, 2019

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

The marks are given as a guide only; the final marking scheme might differ *slightly* from the marks provided here.

Exercises 1 and 2 are common with PCC2.

Exercise 1 (*5.5 marks*).

1. Determine the radius of convergence of the power series

$$\sum_{n=0}^{+\infty} e^n z^{2n}.$$

2. Let $\sum_n a_n z^n$ and $\sum_n b_n z^n$ be two power series of respective radii of convergence R_a and R_b. If $R_a = 1$ and the radius of convergence of the power series $\sum_n (a_n + b_n) z_n$ is 3, what can you say about R_b?

3. Determine the power series expansion of the expression $x\ln(2+x)$, and specify its radius of convergence.

4. For each of the following power series, determine their radii of convergence R, and express their sum in terms of usual functions on their open interval of convergence $(-R, R)$.

a) $\displaystyle\sum_{n=0}^{+\infty} \frac{z^n}{n+1}$ *(useful for exercise 2).*

b) $\displaystyle\sum_{n=1}^{+\infty} \frac{z^{2n}}{(n-1)!}.$

Exercise 2 (*5 marks*).

1. Show that

$$\forall x \in (-1,1),\ \frac{x^3}{(1-x)^2} = \sum_{n=3}^{+\infty} (n-2)x^n.$$

2. Define the following differential equation:

$$\text{(E)} \qquad x^2 y''(x) - 2y(x) = \frac{x^3}{(1-x)^2}.$$

a) Determine all the solutions of Equation (E) that possess a power series expansion. *Hint: there's an infinite number of such solutions.*

b) Sketch the graph of the solutions of Equation (E) you have obtained; there are several cases to handle.

c) Express the solutions of Equation (E) you have obtained in terms of usual functions. *You may use the result of Question 4a from Exercise 1.*

Exercise 3 (*4.5 marks*). In this exercise you may use, without justifications, the well-known limit:

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = \mathrm{e}.$$

1. Let $\alpha \in \mathbb{R}_+^* \setminus \{1/\mathrm{e}\}$. Determine the convergence of the following series:

$$\sum_{n=0}^{+\infty} \frac{(n\alpha)^n}{n!}.$$

2. The case $\alpha = 1/\mathrm{e}$. Define the sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}^*}$ as

$$\forall n \in \mathbb{N},\ u_n = \frac{n^n}{\mathrm{e}^n n!}, \qquad \text{and} \qquad \forall n \in \mathbb{N}^*,\ v_n = \ln(nu_n).$$

The goal of this question is to study the nature of the series $\sum_n u_n$.

a) Briefly explain why the ratio test fails.

b) Show that

$$\frac{u_{n+1}}{u_n} \underset{n\to+\infty}{=} 1 - \frac{1}{2n} + o\left(\frac{1}{n}\right).$$

c) i) Show that:

$$v_{n+1} - v_n \underset{n\to+\infty}{\sim} \frac{n+1}{n}\frac{u_{n+1}}{u_n} - 1 \underset{n\to+\infty}{\sim} \frac{1}{2n}.$$

ii) Deduce the nature of the series $\sum_n (v_{n+1} - v_n)$.

iii) Deduce, using the fact that the series $\sum_n (v_{n+1} - v_n)$ is a telescopic series, that $\lim\limits_{n\to+\infty} v_n = +\infty$.

d) Deduce the nature of the series $\sum_n u_n$.

Exercise 4 (*4 marks*). Define the mapping q as:

$$q: \begin{array}{ccc} \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ (x,y,z) & \longmapsto & 2xy + y^2 - 4yz + 2z^2. \end{array}$$

Define the basis $\mathscr{C} = (u_1, u_2, u_3)$ of $\mathbb{R}^3$, where

$$u_1 = (1,1,1), \qquad u_2 = (0,1,1), \qquad u_3 = (0,0,1).$$

1. Give the matrix $A = [q]_{\mathrm{std}}$ of q in the standard basis of $\mathbb{R}^3$ (no justifications required).
2. Give an explicit form of the polar form φ of q (no justifications required).
3. By using the definition of the matrix of a bilinear form, determine the matrix $A' = [q]_{\mathscr{C}} = [\varphi]_{\mathscr{C}}$ of q (and of φ) in the basis $\mathscr{C}$.
4. By using the change of basis formula, determine A' again.
5. Determine $F = \{u_1 + u_2\}^{\perp}$, the orthogonal (with respect to φ) of the vector $u_1 + u_2$.

Exercise 5 (*1 mark*). Let E be a vector space over $\mathbb{R}$ and let φ be a bilinear form on E.

1. Let $f: E \to E$ be an endomorphism of E. Check that the mapping

$$\psi: \begin{array}{ccc} E\times E & \longrightarrow & \mathbb{R} \\ (u,v) & \longmapsto & \varphi(u, f(v)) \end{array}$$

is a bilinear form on E.

2. If E is finite-dimensional and $\mathscr{B}$ is a basis of E, express the matrix $B = [\psi]_{\mathscr{B}}$ of ψ in the basis $\mathscr{B}$ in terms of the matrix $A = [\varphi]_{\mathscr{B}}$ of φ in the basis $\mathscr{B}$ and of the matrix $M = [f]_{\mathscr{B}}$ of f in the basis $\mathscr{B}$.