INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Solution of Math Test #5

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Exercise 1.

1. We first determine the critical points of f on $\mathbb{R}^2$: let $(x,y)\in\mathbb{R}^2$. Then

$$\begin{aligned}(x,y) \text{ is a critical point of } f &\iff \begin{cases}3y-3x^2=0\\3x-3y^2=0\end{cases}\\&\iff \begin{cases}y=x^2\\x-x^4=0\end{cases}\\&\iff \begin{cases}y=x^2\\x(1-x^3)=0\end{cases}\\&\iff \begin{cases}y=0\\x=0\end{cases}\text{ or }\begin{cases}y=1\\x=1.\end{cases}\end{aligned}$$

Hence f possesses two critical points on $\mathbb{R}^2$, namely, $(0,0)$ and $(1,1)$.

We now study the nature of these critical points. The Hessian matrix of f at a point $(x,y)\in\mathbb{R}^2$ is:

$$H_{(x,y)}f=\begin{pmatrix}-6x & 3\\ 3 & -6y\end{pmatrix}.$$

- At $(0,0)$:

$$H_{(0,0)}f=\begin{pmatrix}0 & 3\\ 3 & 0\end{pmatrix}.$$

  Since $\det H_{(0,0)}f=-9<0$, we conclude that $\operatorname{sign} H_{(0,0)}f=(1,1)$, hence f has a saddle point at $(0,0)$ hence f has no local extreme value at $(0,0)$.

- At $(1,1)$:

$$H_{(1,1)}f=\begin{pmatrix}-6 & 3\\ 3 & -6\end{pmatrix}.$$

  Since $\det H_{(1,1)}f=27>0$ and $\operatorname{tr} H_{(1,1)}f=-12<0$, we conclude that $\operatorname{sign} H_{(1,1)}f=(0,2)$, hence f has a local maximum at $(1,1)$.

Conclusion: f as a unique local maximum at $(1,1)$ and no local minimum.

2. Since $f(x,0)=-x^3+8 \underset{x\to+\infty}{\longrightarrow} = -\infty$, f is not bounded from below, and since $f(x,0)=-x^3+8 \underset{x\to-\infty}{\longrightarrow} = +\infty$, f is not bounded from above. We conclude that f has no global extreme values on $\mathbb{R}^2$.

3. The square S is a closed and bounded set, and f is continuous on S hence, by the Extreme Value Theorem, the restriction of f to S is bounded and attains its bounds. We study the extreme values of f on ∂S, that we divide in four parts:

- On the lower side of S, namely $[0,2]\times\{0\}$: define the function

$$\begin{aligned}g\ :\ [0,2] &\longrightarrow \mathbb{R}\\ x &\longmapsto f(x,0)=-x^3+8.\end{aligned}$$

  Clearly, g is decreasing, hence

$$\min g=g(2)=0 \qquad\text{and}\qquad \max g=g(0)=8.$$

  The minimum corresponds to the point $(2,0)$ and the maximum to the point $(0,0)$.

- On the upper side of S, namely $[0,2]\times\{1\}$: define the function

$$\begin{aligned}g\ :\ [0,2] &\longrightarrow \mathbb{R}\\ x &\longmapsto f(x,2)=-x^3+6x.\end{aligned}$$

For $x \in [0,2]$, $g'(x) = -3x^2 + 6$, hence g is increasing on $[0, \sqrt{2}]$ and decreasing on $[\sqrt{2}, 2]$, hence

$$\min g = \min\{g(0), g(2)\} = 0 \qquad \text{and} \qquad \max g = g(\sqrt{2}) = 4\sqrt{2}.$$

The minimum corresponds to the point $(0,2)$ and the maximum to the point $(\sqrt{2}, 2)$.

- On the left side of S, namely $\{0\} \times [0,2]$: define the function

$$g : \begin{array}{ccc} [0,2] & \longrightarrow & \mathbb{R} \\ y & \longmapsto & f(0,y) = -y^3 + 8, \end{array}$$

hence (similar to the lower side):

$$\min g = 0 \qquad \text{and} \qquad \max g = 8.$$

The minimum corresponds to the point $(0,2)$ and the maximum to the point $(0,0)$.

- On the right side of S, namely $\{2\} \times [0,2]$: define the function

$$g : \begin{array}{ccc} [0,2] & \longrightarrow & \mathbb{R} \\ y & \longmapsto & f(2,y) = -y^3 + 6y, \end{array}$$

hence (similar to the upper side):

$$\min g = 0 \qquad \text{and} \qquad \max g = 4\sqrt{2}.$$

The minimum corresponds to the point $(2,0)$ and the maximum to the point $(2, \sqrt{2})$.

At the critical point $(1,1)$, the value of f is: $f(1,1) = 9$ (which is greater than the maximum of f on ∂S). Conclusion:

$$\min_S f = 0 \qquad \text{and} \qquad \max_S f = 9,$$

and the minimum is attained at $(0,0)$ and the maximum at $(1,1)$.

Exercise 2.

1. The matrix of q in the standard basis of $\mathbb{R}^2$ is:

$$A = [q]_{\text{std}} = \begin{pmatrix} 4 & -\sqrt{3} \\ -\sqrt{3} & 2 \end{pmatrix}.$$

The characteristic polynomial of A is:

$$\chi_A(X) = X^2 - 6X + 5 = (X-1)(X-5).$$

Hence the eigenvalues of A are 1 and 5 (both of multiplicity 1). The equation of the eigenspace of A associated with 1 is:

$$E_1 \colon 3x - \sqrt{3}y = 0,$$

and we may choose

$$X_1 = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$$

as an eigenvector of A associated with 1. We set

$$x_1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

Notice that $\|x_1\|_2 = 1$. To obtain an eigenvector of A associated with 5, we can repeat the procedure, or observe that since A is a real symmetric matrix, E_5 is orthogonal to E_1, hence

$$X_5 = \begin{pmatrix} -\sqrt{3}/2 \\ 1/2 \end{pmatrix}$$

is an eigenvector of A associated with 5. We set

$$x_5 = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right).$$

Notice that $\|x_5\|_2 = 1$. We now conclude that the family $\mathscr{B} = (x_1, x_5)$ is an orthonormal basis of $\mathbb{R}^2$ (with respect to the standard dot product) Moreover, since

$$[q]_{\mathscr{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

is diagonal, we conclude that $\mathscr{B}$ is an orthogonal family with respect to φ.

2. Let $(x, y) \in \mathbb{R}^2$ and let x', y' be the coefficients of the coordinates of (x, y) in the basis $\mathscr{B}$:

$$\big[(x, y)\big]_{\mathscr{B}} = \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

Then

$$(x, y) \in (C) \iff q(x, y) = 25 \iff (x')^2 + 5(y')^2 = 25.$$

We can now plot (C) in the frame $(0, \mathscr{B})$: see Figure 9. And we deduce (C) in the standard frame $(0, e_x, e_y)$: see Figure 10.

Exercise 3.

1. Since A is a real symmetric matrix, there exists an orthogonal matrix P and a diagonal matrix D such that $A = PD\,{}^tP$ (Spectral Theorem).

2. 3 is an eigenvalue of A since

$$\operatorname{rk}(A - 3I_3) = \operatorname{rk}\begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{pmatrix} = 1.$$

By the Rank–Nullity Theorem, we deduce that 3 has multiplicity 2. Using the trace, we conclude that the other eigenvalue of A is 9 (of multiplicity 1). Since A is a real symmetric matrix, we know that $E_3 \perp E_9$, and we use this fact in the sequel.

An equation of E_3 is

$$E_3\colon x - y - 2z = 0$$

from which we deduce that $X_9 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ is an eigenvector of A associated with the eigenvalue 9. We choose a vector of E_3, say

$$X_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

and using a cross product, we find another eigenvector of A associated with 3:

$$X_3' = X_9 \times X_3 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}.$$

To obtain P, we divide X_3, X_3' and X_9 by their norm and stack them together:

$$P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1\sqrt{3} & -1\sqrt{6} \\ 0 & 1/\sqrt{3} & -2\sqrt{6} \end{pmatrix}.$$

And if we set

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

we have $A = PD\,{}^tP$.

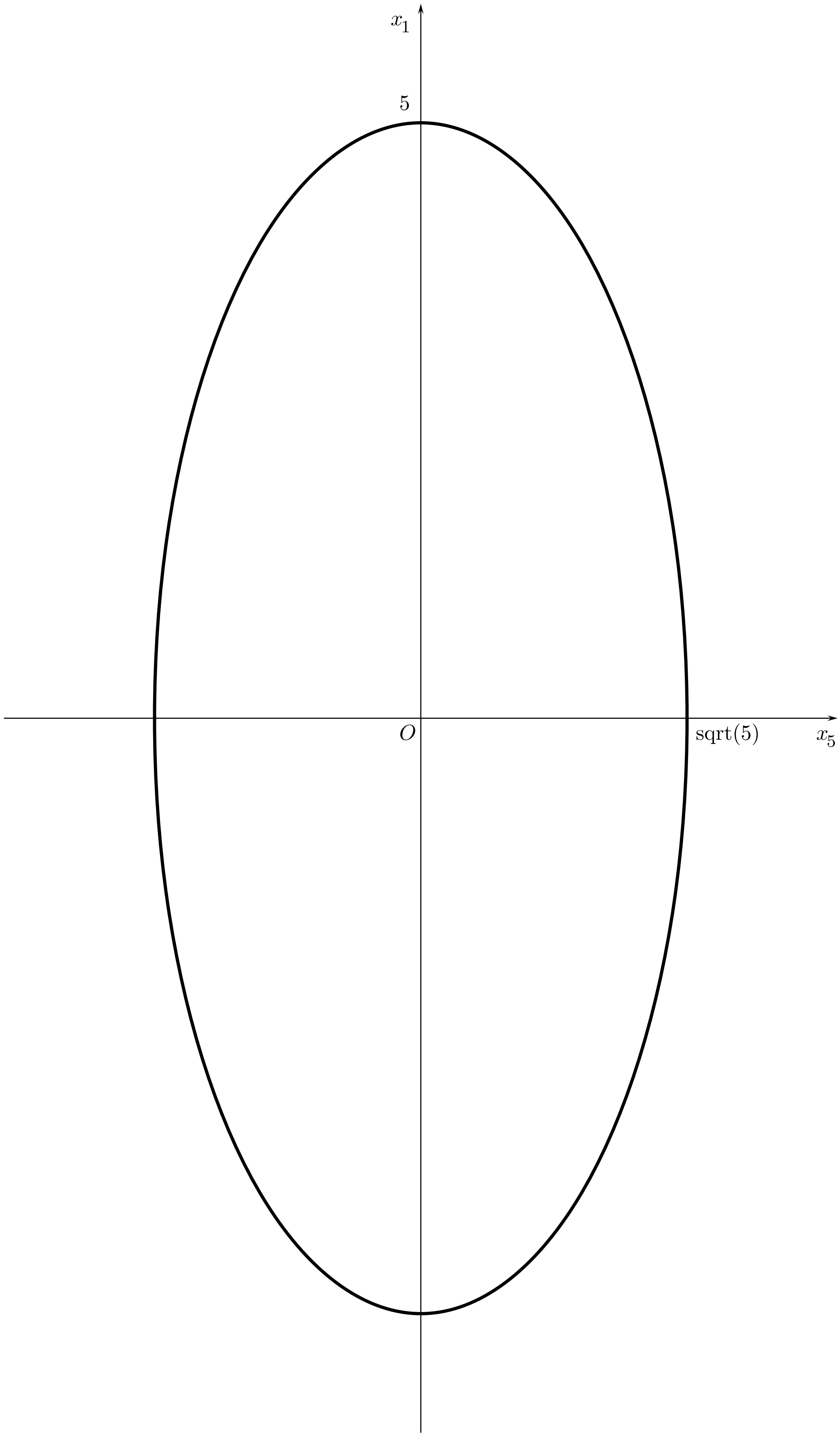

Figure 9. The conic (C) of Exercise 2 in the frame $(O, \mathscr{B})$.

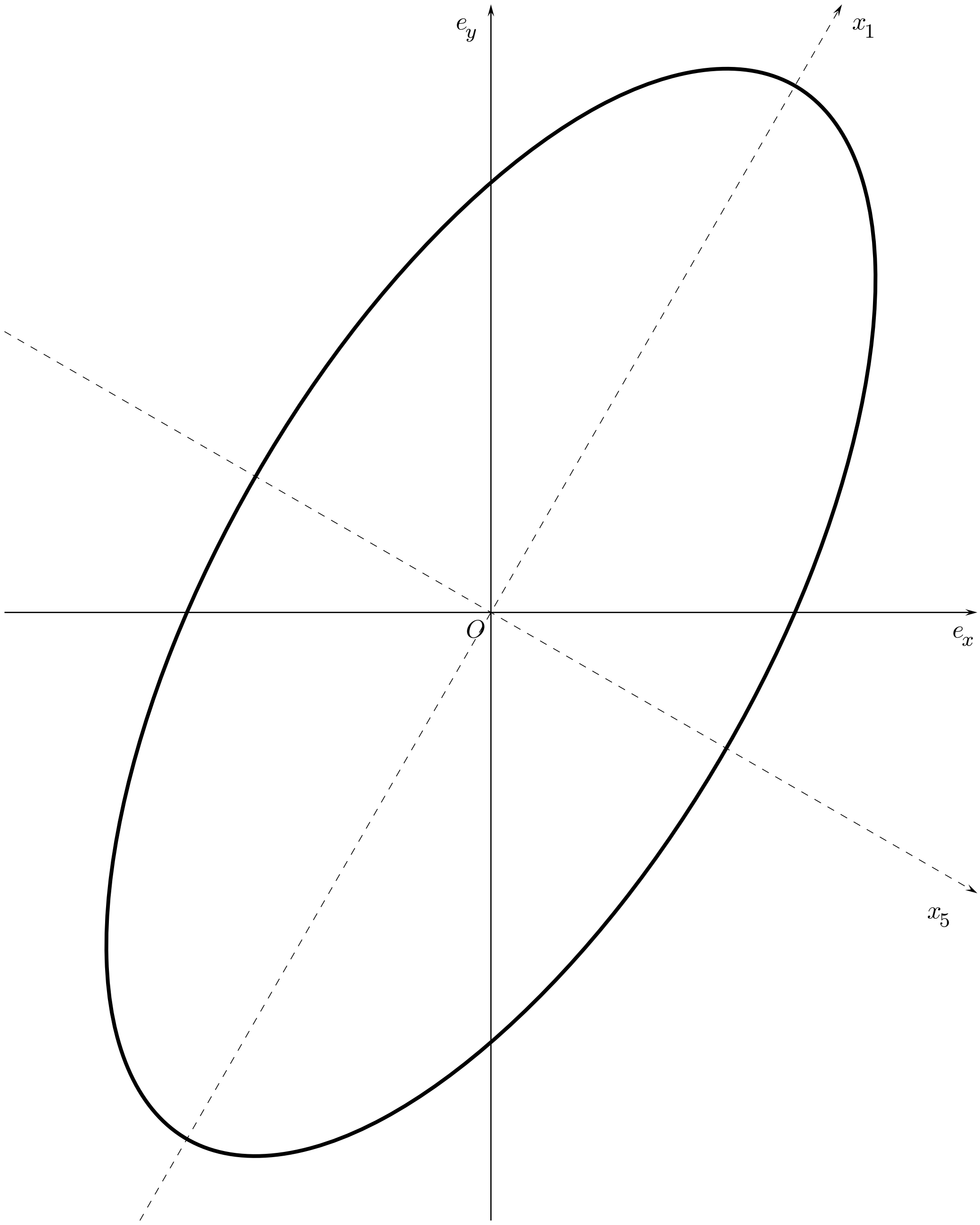

Figure 10. The conic (C) of Exercise 2 in the standard frame (O, e_x, e_y).

3. a) Since the signature of q is $(3,0)$, we conclude that q is positive definite, hence $\sqrt{q}$ is a norm on $\mathbb{R}^3$. We then recognize S as the ball of $\mathbb{R}^3$ (with respect to the norm $\sqrt{q}$) of radius 2; hence S is closed and bounded.

   b) Since $[q]_{\mathscr{B}} = D \neq I_3$, we conclude that $\mathscr{B}$ is not an orthonormal family with respect to φ.

   c) First recall the relation between $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $X' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$:

$$X = PX'.$$

   Then:

$$x^2 + y^2 + z^2 = {}^tXX = {}^t(PX')PX' = {}^tX'\,{}^tPPX' = {}^tX'X' = (x')^2 + (y')^2 + (z')^2.$$

   d) Let $(x, y, z) \in \mathbb{R}^3$ and define X and X' as above, i.e.,

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{and} \qquad X' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = {}^tPX.$$

Then

$$(x,y,z) \in S \iff 3(x')^2 + 3(y')^2 + 9(z')^2 \leq 4.$$

Hence, if we define

$$S' = \{(x',y',z') \in \mathbb{R}^3 \mid 3(x')^2 + 3(y')^2 + 9(z')^2 \leq 4\}$$

we have:

$$\sup_{(x,y,z)\in S} x^2 + y^2 + z^2 = \sup_{(x',y',z')\in S'} (x')^2 + (y')^2 + (z')^2.$$

Let $(x',y',z') \in S'$. Then:

$$(x')^2 + (y')^2 + (z')^2 = \frac{1}{3}\big(3(x')^2 + 3(y')^2 + 3(z')^2\big) \leq \frac{1}{3}\big(3(x')^2 + 3(y')^2 + 9(z')^2\big) \leq \frac{4}{3}.$$

At this point we can conclude that

$$\sup_{(x',y',z')\in S'} (x')^2 + (y')^2 + (z')^2 \leq \frac{4}{3}.$$

Now since $(2/\sqrt{3}, 0, 0) \in S'$,

$$\sup_{(x',y',z')\in S'} (x')^2 + (y')^2 + (z')^2 \geq \left(\frac{2}{\sqrt{3}}\right)^2 + 0^2 + 0^2 = \frac{4}{3},$$

and we conclude that the value of this supremum is $4/3$.

Exercise 4.

1. a) See Figure 11.

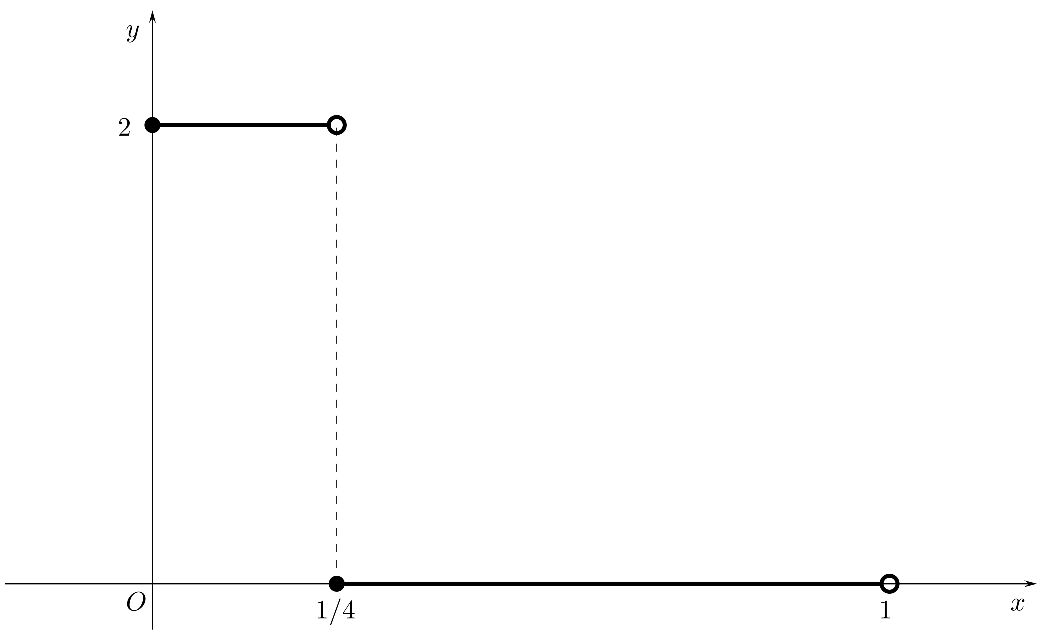

Figure 11. Graph of φ_1 of Exercise 4.

b)

$$\|\varphi_1\|^2 = \int_0^1 \varphi_1(t)^2 \,\mathrm{d}t = \int_0^{1/4} 4 \,\mathrm{d}t = 1,$$

hence $\|\varphi_1\| = 1$.

$$\langle \varphi_1, \varphi_2 \rangle = \int_0^1 \varphi_1(t)\varphi_2(t) \,\mathrm{d}t = 0$$

since $\forall t \in [0,1),\ \varphi_1(t)\varphi_2(t) = 0$.

2. Clearly,

$$\psi_1 = \frac{1}{\sqrt{2}}(\varphi_1 + \varphi_2) \qquad \text{and} \qquad \psi_2 = \frac{1}{\sqrt{2}}(\varphi_3 + \varphi_4)$$

hence $\psi_1, \psi_2 \in E$.

3. We notice that (ψ_1, ψ_2) is an orthonormal basis of G hence:

$$g = \langle f, \psi_1 \rangle \psi_1 + \langle f, \psi_2 \rangle \psi_2.$$

Since $\mathscr{B}$ is an orthonormal family, we obtain:

$$c_1 = \langle f, \psi_1 \rangle = \frac{1}{\sqrt{2}}\big(\langle 5\varphi_1 - \varphi_2 + 2\varphi_3 + 4\varphi_4, \varphi_1 + \varphi_2 \rangle = \frac{1}{\sqrt{2}}(5-1) = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

and

$$c_2 = \langle f, \psi_2 \rangle = \frac{1}{\sqrt{2}}\big(\langle 5\varphi_1 - \varphi_2 + 2\varphi_3 + 4\varphi_4, \varphi_3 + \varphi_4 \rangle = \frac{1}{\sqrt{2}}(2+4) = \frac{6}{\sqrt{2}} = 3\sqrt{2}.$$

4. See Figure 12.

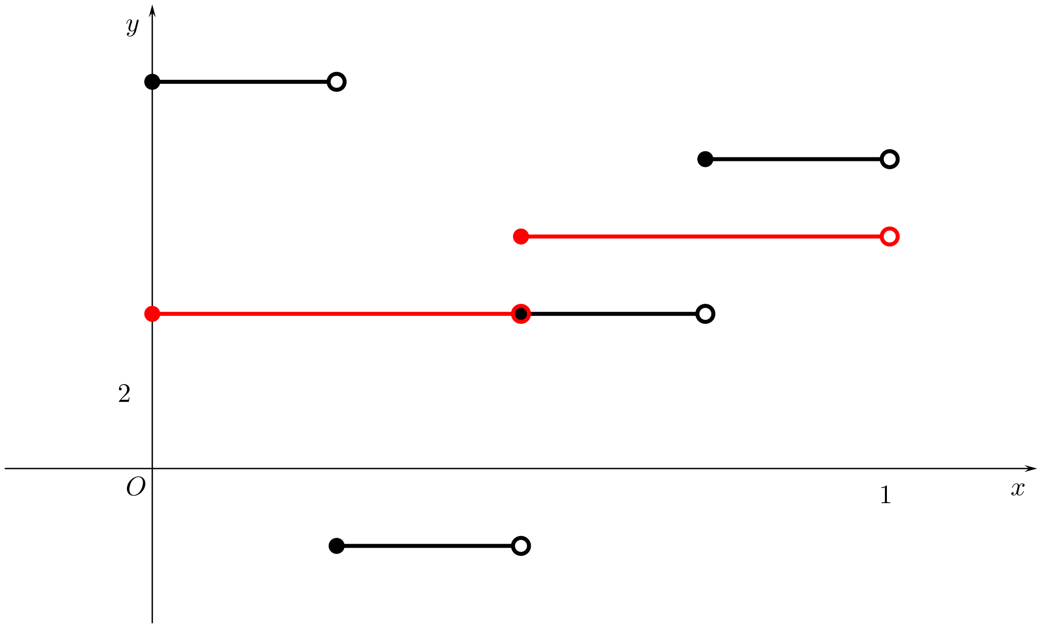

Figure 12. Graph of f (in black) and of g (in red) of Exercise 4.

5. a) Since

$$\inf_{(a_1,a_2)\in\mathbb{R}^2} \big\| f - (a_1\phi_1 + a_2\phi_2) \big\| = \| f - g \|,$$

g is the best approximation (in the sense of $\|\cdot\|$) of f in G. Hence the possible loss of information is minimum when we transmit g.

b) • If we choose $h = \psi_3$, then

$$p_H(f) = g + \langle f, \psi_3 \rangle \psi_3 = g + \frac{1}{\sqrt{2}}(-1-5)\psi_3 = g - 3\sqrt{2}\psi_3.$$

We know that $f - p_H(f)$ and $p_H(f)$ are orthogonal with respect to $\langle \cdot, \cdot \rangle$ hence, by the Pythagorean Theorem (and since (ψ_1, ψ_2, ψ_3) is an orthonormal basis):

$$\| f - p_H(f) \|^2 = \| f \|^2 - \| p_H(f) \|^2 = 46 - (8 + 18 + 18) = 46 - 44 = 2.$$

• If we choose $h = \psi_4$, then

$$p_H(f) = g + \langle f, \psi_4 \rangle \psi_4 = g + \frac{1}{\sqrt{2}}(4-2)\psi_3 = g + 2\sqrt{2}\psi_3.$$

Again,

$$\| f - p_H(f) \|^2 = \| f \|^2 - \| p_H(f) \|^2 = 46 - (8 + 18 + 8) = 46 - 34 = 10.$$

The distance between f and $p_H(f)$ is smaller in the first case, hence it's better to choose $h = \psi_3$.