SCAN 2 — S2 — Math Test #5 — 2h

June 17, 2019

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

The marks are given as a guide only; the final marking scheme might differ *slightly* from the marks provided here.

Exercises 1 and 4 are common with PCC2.

Exercise 1 (*5.5 marks*)**.** We define the function f as:

$$f : \begin{array}{rcl} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x, y) & \longmapsto & 3xy - x^3 - y^3 + 8. \end{array}$$

1. Determine the local extreme values of f on $\mathbb{R}^2$.
2. Does f possess global extreme values on $\mathbb{R}^2$?
3. Justify that f possesses global extreme values on the square $S = [0, 2] \times [0, 2]$ and determine them.

Exercise 2 (*3.5 marks*)**.** Let q be the quadratic form on $\mathbb{R}^2$ defined by:

$$q : \begin{array}{rcl} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x, y) & \longmapsto & 4x^2 - 2\sqrt{3}xy + 2y^2, \end{array}$$

and let φ be the polar form of q. Let (C) be the conic defined by:

$$(C) \qquad q(x, y) = 25.$$

1. Find an orthonormal (with respect to the standard dot product of $\mathbb{R}^2$) basis $\mathscr{B}$ of $\mathbb{R}^2$, that is also orthogonal with respect to φ.
2. Sketch (C) in the frame $(O, \mathscr{B})$ and in the standard frame (O, e_x, e_y).

Exercise 3 (*4.5 marks*)**.** Let

$$A = \begin{pmatrix} 4 & -1 & -2 \\ -1 & 4 & 2 \\ -2 & 2 & 7 \end{pmatrix}.$$

1. Explain why there exists an orthogonal matrix P and a diagonal matrix D such that $A = PD\,{}^tP$.
2. Determine such matrices P and D.

   We denote by q the quadratic form on $\mathbb{R}^3$ such that $[q]_{\text{std}} = A$, by φ the polar form of q, and by $\mathscr{B}$ the basis of $\mathbb{R}^3$ such that $[\mathscr{B}]_{\text{std}} = P$.
3. Let
$$S = \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, q(x, y, z) \le 4\right\}.$$
   a) Is the set S a closed and bounded set?
   b) Is the basis $\mathscr{B}$ an orthonormal basis with respect to φ?
   c) Let $(x, y, z) \in \mathbb{R}^3$ and define x', y', z' as its coordinates in the basis $\mathscr{B}$:
$$\big[(x, y, z)\big]_{\mathscr{B}} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}.$$
   Show that $x^2 + y^2 + z^2 = (x')^2 + (y')^2 + (z')^2$.
   d) Compute the value of
$$\sup_{(x,y,z)\in S} x^2 + y^2 + z^2.$$

Exercise 4 (*6.5 marks*)**.** For $i \in \{1, 2, 3, 4\}$ We define the functions

$$\begin{array}{rccl} \varphi_i & : & [0,1) & \longrightarrow & \mathbb{R} \\ & & t & \longmapsto & \begin{cases} 2 & \text{if } \dfrac{i-1}{4} \leq t < \dfrac{i}{4} \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

We define $E = \mathrm{Span}\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$. We define the bilinear form $\langle \cdot, \cdot \rangle$ on E as:

$$\forall f, g \in E,\ \langle f, g \rangle = \int_0^1 f(t)g(t)\,\mathrm{d}t.$$

You're given that $\langle \cdot, \cdot \rangle$ is an inner product on E. We denote by $\|\cdot\|$ the associated Euclidean norm.

1. a) Sketch the graph of φ_1.
   b) Compute $\|\varphi_1\|$ and $\langle \varphi_1, \varphi_2 \rangle$.
   You're given that $\mathscr{B}_\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ is an orthonormal basis of E.

2. We define the functions ψ_1 and ψ_2 as:

$$\begin{array}{rccl} \psi_1 & : & [0,1) & \longrightarrow & \mathbb{R} \\ & & t & \longmapsto & \begin{cases} \sqrt{2} & \text{if } 0 \leq t < \dfrac{1}{2} \\ 0 & \text{otherwise} \end{cases} \end{array} \qquad \begin{array}{rccl} \psi_2 & : & [0,1) & \longrightarrow & \mathbb{R} \\ & & t & \longmapsto & \begin{cases} \sqrt{2} & \text{if } \dfrac{1}{2} \leq t < 1 \\ 0 & \text{otherwise} \end{cases} \end{array}$$

   and we define

$$G = \mathrm{Span}\{\psi_1, \psi_2\}.$$

   a) Show that ψ_1 and ψ_2 belong to E.
   b) Let

$$f = 5\varphi_1 - \varphi_2 + 2\varphi_3 + 4\varphi_4.$$

   Compute the orthogonal projection $g = c_1\psi_1 + c_2\psi_2$ of f on G.
   c) Sketch, on the same figure, the graphs of the functions f and g.

3. The function f represents a signal that we want to transmit through a network. We could transmit the coefficients $(5, -1, 2, 4)$, corresponding to the coordinates of f in $\mathscr{B}_\varphi$. For the sake of efficiency, but with some possible loss of information, we decide to only transmit the coefficients (c_1, c_2), corresponding to the coordinates of g in the basis (ψ_1, ψ_2).
   a) Explain why the coefficients c_1 and c_2 (corresponding to the coordinates of g in the basis (ψ_1, ψ_2)) represent a good option for the transmission of an approximation of f.
   b) We decide to transmit three values (instead of two), by considering the orthogonal projection of f on

$$H = \mathrm{Span}\{\psi_1, \psi_2, h\},$$

   where we have the choice for h: either $h = \psi_3$ or $h = \psi_4$ where

$$\psi_3 = \frac{1}{\sqrt{2}}(\varphi_2 - \varphi_1) \qquad \text{or} \qquad \psi_4 = \frac{1}{\sqrt{2}}(\varphi_4 - \varphi_3).$$

   You're given that (ψ_1, ψ_2, ψ_3) is an orthonormal basis of $\mathrm{Span}\{\psi_1, \psi_2, \psi_3\}$ and (ψ_1, ψ_2, ψ_4) is an orthonormal basis of $\mathrm{Span}\{\psi_1, \psi_2, \psi_4\}$. How should you choose h so that $f = 5\varphi_1 - \varphi_2 + 2\varphi_3 + 4\varphi_4$ is best transmitted? Compute the orthogonal projection of f on H in this case.