INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Solution of Math Test #1

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Exercise 1.

1. The function $t \mapsto \dfrac{\sin(t)}{1+t^2}$ is continuous on $[0,+\infty)$, hence I is improper at $+\infty$. Clearly,

$$\forall t \in [1,+\infty),\ \left|\frac{\sin(t)}{1+t^2}\right| \leq \frac{1}{t^2},$$

hence I converges absolutely, hence I converges.

2. The function $t \mapsto \dfrac{1}{1-\sqrt{t}}$ is continuous on $[0,1)$, hence J is improper at 1^-. Now,

$$\forall t \in [0,1),\ \frac{1}{1-\sqrt{t}} = \frac{1+\sqrt{t}}{1-t} \underset{t\to 1^-}{\sim} \frac{2}{1-t} > 0.$$

By Riemann at a finite point, we know that the improper integral $\int_0^1 \frac{2\mathrm{d}t}{1-t}$ diverges at 1^-, hence we conclude, by the equivalent test, that J is divergent.

Exercise 2.

1. The function ln is continuous on $(0,1]$, hence K is improper at 0^+. Let $A \in (0,1)$. Then:

$$\begin{aligned}\int_A^1 \ln(t)\,\mathrm{d}t &= \big[t\ln(t) - t\big]_{t=A}^{t=1}\\ &= -1 - A\ln(A) - A\\ &\underset{A\to 0^+}{\longrightarrow} -1 \in \mathbb{R},\end{aligned}$$

hence K converges and $K = -1$.

2. For $x \geq 0$, the function $t \mapsto \ln\big(1+t^x\big)$ is continuous on $[0,1]$, hence $F(x)$ is a definite integral of a continuous function on a closed and bounded interval, hence $F(x)$ is well-defined.

3. Let $x < 0$. The function $t \mapsto \ln\big(1+t^x\big)$ is continuous on $(0,1]$, hence the improper integral $F(x)$ is improper at 0^+.

   For $t \in (0,1]$,

$$\ln\big(1+t^x\big) = \ln\Big(t^x\big(t^{-x}+1\big)\Big) = \ln\big(t^x\big) + \ln\big(1+t^{-x}\big).$$

   Since $x < 0$, we have $\ln\big(t^x\big) \underset{t\to 0^+}{\longrightarrow} +\infty$ and $\ln\big(1+t^{-x}\big) \underset{t\to 0^+}{\longrightarrow} 0$, hence

$$\ln\big(1+t^x\big) \underset{t\to 0^+}{\sim} \ln\big(t^x\big) = x\ln(t) > 0.$$

   By the equivalent test and by Question 1, we conclude that the improper integral $F(x)$ is convergent.

   Hence $F(x)$ is defined for $x \geq 0$ (Question 2) and for $x < 0$ (this question), hence $F(x)$ is defined for $x \in \mathbb{R}$.

4. Let $t \in (0,1]$ and $x \in (-\infty,0)$. Then $1+t^x \geq t^x$ hence $\ln\big(1+t^x\big) \geq \ln\big(t^x\big) = x\ln(t)$.

   From the previous inequality we conclude:

$$\forall x \in (-\infty,0),\ F(x) = \int_0^1 \ln\big(1+t^x\big)\,\mathrm{d}t \geq x\int_0^1 \ln(t)\,\mathrm{d}t = -x.$$

   Hence, by the Squeeze Theorem, $\lim\limits_{x\to-\infty} F(x) = +\infty$.

5. Let $A \in (0,1]$ and $x \in (0,+\infty)$. We use the substitution $z = t^x$, which yields $t = z^{1/x}$ hence $\mathrm{d}t = \dfrac{z^{1/x-1}\mathrm{d}z}{x}$:

$$\int_A^1 \ln\big(1+t^x\big)\,\mathrm{d}t = \frac{1}{x}\int_{A^x}^1 \ln(1+z)z^{1/x-1}\mathrm{d}z \underset{A\to 0^+}{\longrightarrow} \frac{1}{x}\int_0^1 \ln(1+z)z^{1/x-1}\,\mathrm{d}z,$$

   since $A^x \underset{A\to 0^+}{\longrightarrow} 0$ since $x > 0$.

6. Let $x \in \mathbb{R}_+^*$ and let $z \in (0,1]$. Then, by the given inequality,

$$\ln(1+z)z^{1/x-1} \leq z^{1/x},$$

hence, by integrating with respect to z from 0 to 1 (with $0 < 1$) yields

$$F(x) \leq \frac{1}{x}\int_0^1 z^{1/x}\,\mathrm{d}z = \frac{1}{x}\left[xz^{1+1/x}\right]_{z=0}^{z=1} = \frac{1}{1+x}.$$

Clearly, $F(x) \geq 0$ and hence $0 \leq F(x) \leq \dfrac{1}{1+x}$ and we conclude, by the Squeeze Theorem, that $\lim\limits_{x\to+\infty} F(x) = 0$.

Exercise 3.

1. Let $n \in \mathbb{N}^*$. The function $x \mapsto \dfrac{1}{\left(1+x^2\right)^n}$ is continuous on $[0,+\infty)$, hence the improper integral I_n is improper at $+\infty$. Now

$$\frac{1}{\left(1+x^2\right)^n} \underset{x\to+\infty}{\sim} \frac{1}{x^{2n}} > 0,$$

and by Riemann at $+\infty$ (with $\alpha = 2n \geq 2 > 1$) we know that the improper integral $\int_1^{+\infty} \dfrac{\mathrm{d}x}{x^{2n}}$ converges. Hence, by the equivalent test, I_n converges.

2. Let $A \in \mathbb{R}_+^*$. By an integration by parts with $u(x) = \dfrac{1}{\left(1+x^2\right)^n}$ and $v'(x) = 1$ and hence $u'(x) = -\dfrac{2nx}{\left(1+x^2\right)^{n+1}}$ and $v(x) = x$, we have:

$$\int_0^A \frac{\mathrm{d}x}{\left(1+x^2\right)^n} = \left[\frac{x}{\left(1+x^2\right)^n}\right]_{x=0}^{x=A} - \int_0^A \frac{-2nx^2}{\left(1+x^2\right)^{n+1}}\,\mathrm{d}x = \frac{A}{\left(1+A^2\right)^n} + 2n\int_0^A \frac{x^2}{\left(1+x^2\right)^{n+1}}\,\mathrm{d}x$$

Hence, by taking the limit as $A \to +\infty$ we obtain (since $n \geq 1$):

$$I_n = \int_0^{+\infty} \frac{\mathrm{d}x}{\left(1+x^2\right)^n} = 0 + 2n\int_0^{+\infty} \frac{x^2}{\left(1+x^2\right)^{n+1}}\,\mathrm{d}x.$$

Now

$$\begin{aligned}
I_n &= 2n\int_0^{+\infty} \frac{x^2}{\left(1+x^2\right)^{n+1}}\,\mathrm{d}x \\
&= 2n\int_0^{+\infty} \left(\frac{1+x^2}{\left(1+x^2\right)^{n+1}} - \frac{1}{\left(1+x^2\right)^{n+1}}\right)\mathrm{d}x \\
&= 2n\int_0^{+\infty} \left(\frac{1}{\left(1+x^2\right)^{n}} - \frac{1}{\left(1+x^2\right)^{n+1}}\right)\mathrm{d}x \\
&= 2nI_n - 2nI_{n+1}
\end{aligned}$$

hence $(1-2n)I_n = -2nI_{n+1}$ and we conclude that $I_n = \dfrac{2n}{2n-1}I_{n+1}$.

3. Let $n \in \mathbb{N}^*$. Then

$$\begin{aligned}
\sum_{k=1}^n I_k &= \sum_{k=1}^n \int_0^{+\infty} \frac{\mathrm{d}x}{\left(1+x^2\right)^k} \\
&= \int_0^{+\infty} \sum_{k=1}^n \frac{1}{\left(1+x^2\right)^k}\,\mathrm{d}x \\
&= \int_0^{+\infty} \sum_{k=1}^n \left(\frac{1}{1+x^2}\right)^k \mathrm{d}x.
\end{aligned}$$

At this point, we want to use the formula for the sum of a geometric progression, which is valid when the ratio $q = \dfrac{1}{1+x^2} \neq 1$, i.e., for $x \neq 0$. Since 0 is in the domain of integration, we have to be a little bit careful: Let $A \in \mathbb{R}_+^*$. Then:

$$\begin{aligned}
\int_A^{+\infty} \sum_{k=1}^{n} \left(\frac{1}{1+x^2}\right)^k \mathrm{d}x &= \int_A^{+\infty} \frac{\dfrac{1}{1+x^2} - \left(\dfrac{1}{1+x^2}\right)^{n+1}}{1 - \dfrac{1}{1+x^2}} \mathrm{d}x \\
&= \int_A^{+\infty} \frac{1 - \left(\dfrac{1}{1+x^2}\right)^n}{x^2} \mathrm{d}x \\
&= \int_A^{+\infty} \frac{1}{x^2}\left(1 - \frac{1}{\left(1+x^2\right)^n}\right) \mathrm{d}x \\
&\underset{A\to 0^+}{\longrightarrow} \int_0^{+\infty} \frac{1}{x^2}\left(1 - \frac{1}{\left(1+x^2\right)^n}\right) \mathrm{d}x.
\end{aligned}$$

4. Let $n \in \mathbb{N}^*$. Since for $x > 0$,

$$\frac{1}{x^2}\left(1 - \frac{1}{\left(1+x^2\right)^n}\right) \geq 0,$$

and since $[1/\sqrt{n}, 1] \subset [0, +\infty)$, we conclude:

$$\int_0^{+\infty} \frac{1}{x^2}\left(1 - \frac{1}{\left(1+x^2\right)^n}\right) \mathrm{d}x \geq \int_{1/\sqrt{n}}^{1} \frac{1}{x^2}\left(1 - \frac{1}{\left(1+x^2\right)^n}\right) \mathrm{d}x$$

Moreover, the function $x \mapsto \left(1 - \dfrac{1}{\left(1+x^2\right)^n}\right)$ is increasing on $\mathbb{R}_+$, hence

$$\forall x \in [1/\sqrt{n}, 1], \quad \left(1 - \frac{1}{\left(1+x^2\right)^n}\right) \geq \left(1 - \frac{1}{\left(1+1/n\right)^n}\right)$$

hence

$$\forall x \in [1/\sqrt{n}, 1], \quad \frac{1}{x^2}\left(1 - \frac{1}{\left(1+x^2\right)^n}\right) \geq \frac{1}{x^2}\left(1 - \frac{1}{\left(1+1/n\right)^n}\right).$$

Now, integrating this inequality from $1/\sqrt{n}$ to 1 yields:

$$\begin{aligned}
\int_{1/\sqrt{n}}^{1} \frac{1}{x^2}\left(1 - \frac{1}{\left(1+x^2\right)^n}\right) \mathrm{d}x &\geq \int_{1/\sqrt{n}}^{1} \frac{1}{x^2}\left(1 - \frac{1}{\left(1+1/n\right)^n}\right) \mathrm{d}x \\
&= \left(1 - \frac{1}{\left(1+1/n\right)^n}\right)\left[-\frac{1}{x}\right]_{x=1/\sqrt{n}}^{x=1} \\
&= \left(\sqrt{n}-1\right)\left(1 - \frac{1}{\left(1+1/n\right)^n}\right),
\end{aligned}$$

hence the result.

5. We know that

$$\lim_{n\to+\infty} \left(1 + \frac{1}{n}\right)^n = \mathrm{e},$$

hence

$$\left(\sqrt{n}-1\right)\left(1 - \frac{1}{\left(1+1/n\right)^n}\right) \underset{n\to+\infty}{\sim} \mathrm{e}\sqrt{n} \underset{n\to+\infty}{\longrightarrow} +\infty$$

hence, by the Squeeze Theorem,

$$\lim_{n\to+\infty} \sum_{k=1}^{n} I_k = +\infty.$$

Exercise 4.

1.
   - Separation property: let $f \in E$ such that $N(f) = 0$, hence

$$\int_0^1 t\,\bigl|f(t)\bigr|\,\mathrm{d}t = 0.$$

   Since the function $t \mapsto t\bigl|f(t)\bigr|$ is *continuous and non-negative*, we conclude that

$$\forall t \in [0,1],\ t\bigl|f(t)\bigr| = 0,$$

   hence

$$\forall t \in (0,1],\ f(t) = 0$$

   (notice the open interval at 0). Since f is continuous, we have:

$$f(0) = \lim_{t \to 0^+} f(t) = \lim_{t \to 0^+} 0 = 0,$$

   and we conclude that $f = 0_E$.

   - Positive homogeneity: let $f \in E$ and let $\lambda \in \mathbb{R}$. Then

$$N(\lambda f) = \int_0^1 t\,\bigl|\lambda f(t)\bigr|\,\mathrm{d}t = |\lambda| \int_0^1 t\,\bigl|f(t)\bigr|\,\mathrm{d}t = |\lambda| N(f).$$

   - Triangle inequality: let $f, g \in E$. Then:

$$\begin{aligned} N(f+g) &= \int_0^1 t\bigl|f(t)+g(t)\bigr|\,\mathrm{d}t \\ &\leq \int_0^1 t\Bigl(\bigl|f(t)\bigr| + \bigl|g(t)\bigr|\Bigr)\,\mathrm{d}t \\ &= \int_0^1 t\bigl|f(t)\bigr|\,\mathrm{d}t + \int_0^1 t\bigl|g(t)\bigr|\,\mathrm{d}t \\ &= N(f) + N(g). \end{aligned}$$

2. Let $f \in E$. Since

$$\forall t \in [0,1],\ t\bigl|f(t)\bigr| \leq \bigl|f(t)\bigr|,$$

   we conclude (by integrating from 0 to 1) that:

$$\int_0^1 t\bigl|f(t)\bigr|\,\mathrm{d}t \leq \int_0^1 \bigl|f(t)\bigr|,$$

   i.e., $N(f) \leq \|f\|_1$.

3. The second statement is correct. Indeed, assume that the sequence $(f_n)_{n \in \mathbb{N}}$ converges to f for the norm $\|\cdot\|_1$. This means that $\lim\limits_{n \to +\infty} \|f_n - f\|_1 = 0$. Since $0 \leq N \leq \|\cdot\|_1$ we conclude, by the Squeeze Theorem, that $\lim\limits_{n \to +\infty} N(f_n - f) = 0$, i.e., that $(f_n)_{n \in \mathbb{N}}$ converges to f for the norm N.

4. a) Let $n \in \mathbb{N}^*$. Then:

$$N(f_n) = \int_0^{1/n} tn\,\mathrm{d}t + \int_{1/n}^1 \mathrm{d}t = \frac{1}{2n} + 1 - \frac{1}{n} = 1 - \frac{1}{2n},$$

   and

$$\|f_n\|_1 = \int_0^{1/n} n\,\mathrm{d}t + \int_{1/n}^1 \frac{\mathrm{d}t}{t} = 1 + \ln(n).$$

   b) We proceed by contradiction: assume that N and $\|\cdot\|_1$ are equivalent. Hence there exists $\alpha > 0$ such that $\alpha\|\cdot\|_1 \leq N$. In particular, for $n \in \mathbb{N}^*$ we have

$$\alpha\|f_n\|_1 = \alpha\bigl(1 + \ln(n)\bigr) \leq N(f_n) = 1 - \frac{1}{2n}.$$

   Taking the limit as $n \to +\infty$ yields (since $\alpha > 0$): $+\infty \leq 1$, which is impossible. Hence the norms N and $\|\cdot\|_1$ are not equivalent.

Exercise 5. Let $(h_x, h_y, h_z) \in \mathbb{R}^3$. Then:

$$\begin{aligned} f(1+h_x, -2+h_y, -2+h_z) &= 3(1+h_x)^2 - (1+h_x)(-2+h_y) - 3(-2+h_y)(-2+h_z) + (-2+h_z)^2 \\ &= -3 \;+\; 8h_x + 5h_y + 2h_z \;+\; 3h_x^2 - h_x h_y - 3h_y h_z + h_z^2 \\ &= f(1,-2,-2) \;+\; 8h_x + 5h_y + 2h_z \;+\; 3h_x^2 - h_x h_y - 3h_y h_z + h_z^2. \end{aligned}$$

We obtained the constant term, a linear term with respect to (h_x, h_y, h_z) and a remainder. Since $\mathbb{R}^3$ is a finite dimensional vector space, the linear term is automatically continuous. To prove that f is differentiable at a_0 we only need to prove that the remainder is a $o\big(\|(h_x, h_y, h_z)\|\big)$: for $h = (h_x, h_y, h_z) \neq (0,0,0)$,

$$\left| \frac{3h_x^2 - h_x h_y - 3h_y h_z + h_z^2}{\|h\|_1} \right| \leq \frac{8\|h\|_1^2}{\|h\|_1} = 8\|h\|_1 \underset{h \to (0,0,0)}{\longrightarrow} 0,$$

where, in the numerator, we used the triangle inequality for the absolute value together with the following useful inequalities:

$$|h_x| \leq \|h\|_1 \qquad\qquad |h_y| \leq \|h\|_1 \qquad\qquad |h_z| \leq \|h\|_1.$$

Notice that we chose the 1-norm, but any other norm would produce the same result as all norms are equivalent on $\mathbb{R}^3$.

Hence f is differentiable at a_0 and:

$$\begin{array}{rccc} \mathrm{d}_{a_0} f \;: & \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ & (h_x, h_y, h_z) & \longmapsto & 8h_x + 5h_y + 2h_z. \end{array}$$