SCAN 2 — S1 — Math Test #1 — 2h

October 21, 2018

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

The marks are given as a guide only; the final marking scheme might differ *slightly* from the marks provided here.

Exercises 1, 2 and 3 are (partly) common with PCC2.

Exercise 1 (*2.5 marks*)**.** Determine the nature of the following improper integrals:

$$I = \int_0^{+\infty} \frac{\sin(t)}{1+t^2}\,\mathrm{d}t \qquad \text{and} \qquad J = \int_1^2 \frac{\mathrm{d}t}{1-\sqrt{t}}.$$

Exercise 2 (*5 marks*)**.**

1. *Preliminary question.* Show that the improper integral

$$K = \int_0^1 \ln(t)\,\mathrm{d}t$$

is convergent, and that $K = -1$.

For $x \in \mathbb{R}$, we define the expression $F(x)$ as follows:

$$F(x) = \int_0^1 \ln\big(1+t^x\big)\,\mathrm{d}t.$$

2. Prove that for all $x \geq 0$, $F(x)$ is well-defined.
3. Prove that for all $x < 0$, one has $\ln\big(1+t^x\big) \underset{t\to 0^+}{\sim} x\ln(t)$, and deduce that $F(x)$ is defined for all $x \in \mathbb{R}$.
4. Show that

$$\forall t \in (0,1],\ \forall x \in (-\infty, 0),\ \ln\big(1+t^x\big) \geq x\ln(t).$$

and deduce that $\lim\limits_{x\to-\infty} F(x) = +\infty$

5. Show that

$$\forall x > 0,\ F(x) = \frac{1}{x}\int_0^1 \ln(1+z)\,z^{\frac{1}{x}-1}\,\mathrm{d}z.$$

6. Deduce that $\lim\limits_{x\to+\infty} F(x) = 0$.

You may use, without any justifications, the following fact:

$$\forall z \in [0,1],\ 0 \leq \ln(1+z) \leq z.$$

Exercise 3 (*5.5 marks*)**.** For $n \in \mathbb{N}^*$ we define the following improper integral:

$$I_n = \int_0^{+\infty} \frac{\mathrm{d}x}{\left(1+x^2\right)^n}.$$

1. Prove that for all $n \in \mathbb{N}^*$, the improper integral I_n is convergent.
2. Show that
$$\forall n \in \mathbb{N}^*,\ I_n = 2n \int_0^{+\infty} \frac{x^2}{\left(1+x^2\right)^{n+1}}\,\mathrm{d}x.$$
Deduce that
$$\forall n \in \mathbb{N}^*,\ I_n = \frac{2n}{2n-1} I_{n+1}.$$
3. Show that
$$\forall n \in \mathbb{N}^*,\ \sum_{k=1}^{n} I_k = \int_0^{+\infty} \frac{1}{x^2}\left(1 - \frac{1}{\left(1+x^2\right)^n}\right)\mathrm{d}x.$$
4. Explain why
$$\forall n \in \mathbb{N}^*,\ \sum_{k=1}^{n} I_k \geq \left(\sqrt{n}-1\right)\left(1 - \frac{1}{\left(1+\frac{1}{n}\right)^n}\right).$$
5. What can you deduce about the following limit? $\lim\limits_{n\to+\infty} \sum\limits_{k=1}^{n} I_k$.

Exercise 4 (*5 marks*)**.** Let $E = C\big([0,1]\big)$ be the real vector space of real-valued continuous functions on $[0,1]$. We define:

$$\begin{array}{rccl} N : & E & \longrightarrow & \mathbb{R}_+ \\ & f & \longmapsto & \displaystyle\int_0^1 t\left|f(t)\right|\mathrm{d}t. \end{array}$$

1. Show that N is a norm on E.
2. Show that $N \leq \|\cdot\|_1$.
3. From the previous equality, you can deduce one the following two statements. Which one is correct?
   i) *If a sequence $(f_n)_{n\in\mathbb{N}}$ of elements of E converges to $f \in E$ for N, then the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f for $\|\cdot\|_1$.*
   ii) *If a sequence $(f_n)_{n\in\mathbb{N}}$ of elements of E converges to $f \in E$ for $\|\cdot\|_1$, then the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f for N.*

   Justify your answer (as concisely as possible).
4. Define the sequence $(f_n)_{n\in\mathbb{N}^*}$ of elements of E as:
$$\forall n \in \mathbb{N}^*,\ \begin{array}{rccl} f_n : & [0,1] & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \begin{cases} n & \text{if } x \in \left[0, \frac{1}{n}\right] \\ \frac{1}{x} & \text{if } x \in \left[\frac{1}{n}, 1\right] \end{cases} \end{array}$$
   a) For $n \in \mathbb{N}^*$, compute $N(f_n)$ and $\|f_n\|_1$.
   b) Deduce that the norms N and $\|\cdot\|_1$ are not equivalent.

Exercise 5 (*2 marks*)**.** Define the following function:

$$\begin{array}{rccl} f : & \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ & (x,y,z) & \longmapsto & 3x^2 - xy - 3yz + z^2. \end{array}$$

Let $a_0 = (1,-2,-2)$. Show that f is differentiable at a_0 and determine $\mathrm{d}_{a_0} f$, the differential of f at a_0.