SCAN 2 — Solution of Math Test #2

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Exercise 1.

1. Let $h \in \mathbb{R}^*$. Then:
$$\frac{f(h,0)-f(0,0)}{h} = 0 \xrightarrow[h\to 0]{} 0,$$
hence $\partial_1 f(0,0)$ exists and $\partial_1 f(0,0) = 0$, and
$$\frac{f(0,h)-f(0,0)}{h} = 0 \xrightarrow[h\to 0]{}= \frac{h}{h} = 1 \xrightarrow[h\to 0]{} 1,$$
hence $\partial_2 f(0,0)$ exists and $\partial_2 f(0,0) = 1$.

2. Let $u = (u_x, u_y) \in \mathbb{R}^2$.
   - If $u_y \neq 0$: let $t \in \mathbb{R}^*$. Then:
   $$\frac{f(tu_x,tu_y)-f(0,0)}{t} = \frac{t^2u_x^2\ln\bigl(|tu_y|\bigr)+tu_y}{t} = tu_x^2\ln\bigl(|tu_y|\bigr)+u_y \xrightarrow[t\to 0]{} u_y,$$
   hence $\nabla_u f(0,0) = u_y$,
   - if $u_y = 0$: let $t \in \mathbb{R}^*$. Then:
   $$\frac{f(tu_x,tu_y)-f(0,0)}{t} = 0 \xrightarrow[t\to 0]{} 0,$$
   hence $\nabla_u f(0,0) = 0 = u_y$.

   We conclude that Proposition (P) is true.

3. Observe that $\lim\limits_{t\to 0}\bigl(t, \mathrm{e}^{-1/t^2}\bigr) = (0,0)$. Now, for $t \in \mathbb{R}^*$,
$$f\bigl(t,\mathrm{e}^{-1/t^2}\bigr) = t^2\ln\bigl(\mathrm{e}^{-1/t^2}\bigr) + \mathrm{e}^{-1/t^2} = -1+\mathrm{e}^{-1/t^2} \xrightarrow[t\to 0]{} -1 \neq f(0,0) = 0,$$
hence, by composition of limits, f is not continuous at $(0,0)$.

4. Since f is not continuous at $(0,0)$, f is not differentiable at $(0,0)$.

5. Let $(x,y) \in \mathbb{R}\setminus D$. Then:
$$\partial_1 f(x,y) = 2x\ln\bigl(|y|\bigr)$$
and
$$\partial_2 f(x,y) = \frac{x^2}{y}+1.$$
Hence,
$$\overrightarrow{\operatorname{grad}}\, f(x,y) = \left(2x\ln\bigl(|y|\bigr), \frac{x^2}{y}+1\right).$$

6. Since $(1,1) \in \mathbb{R}\setminus D$ we can reuse the previous expressions:
$$\partial_1 f(1,1) = 0, \qquad \partial_2 f(1,1) = 2,$$
hence
$$\mathrm{d}_{(1,1)} f = 2e_2'.$$

7. $f(1.01, 1.02) \approx f(1,1) + \mathrm{d}_{(1,1)} f(0.01, 0.02) = 1 + 2\times 0.02 = 1.04.$

8. We compute the second order partial derivatives of f at $(1,1)$:

$$\begin{aligned}
\partial^2_{(1,1)} f(x,y) &= 2\ln(|y|), & \partial^2_{(1,1)} f(1,1) &= 0,\\
\partial^2_{(1,2)} f(x,y) &= 2\frac{x}{y}, & \partial^2_{(1,2)} f(1,1) &= 2,\\
\partial^2_{(2,2)} f(x,y) &= -\frac{x^2}{y^2}, & \partial^2_{(2,2)} f(1,1) &= -1.
\end{aligned}$$

Hence the second order Taylor–Young expansion of f is:

$$f(1+h_x, 1+h_y) \underset{(h_x,h_y)\to(0,0)}{=} 1 + 2h_y + 2h_x h_y - \frac{1}{2}h_y^2 + o\big(\|(h_x,h_y)\|^2\big).$$

Exercise 2.

1. Let $(x,y)\in U$ and let $(u,w)\in U$. Then:

$$\varphi(x,y) = (u,w) \iff \begin{cases} x/y = u \\ x+y = w \end{cases} \iff \begin{cases} x = uy \\ y(1+u) = w \end{cases} \iff \begin{cases} x = uw/(1+u) \\ y = w/(1+u) \end{cases}$$

Moreover, $uw/(1+u)\in\mathbb{R}_+^*$ and $w/(1+u)\in\mathbb{R}_+^*$, hence φ is a bijection and

$$\begin{aligned} \varphi^{-1} : \quad U &\longrightarrow U \\ (u,w) &\longmapsto \left(\frac{uw}{1+u}, \frac{w}{1+u}\right). \end{aligned}$$

We notice (from elementary operations and usual functions) that both φ and φ^{-1} are of class C^∞; since φ is a bijection, we conclude that φ is a C^∞-diffeomorphism.

2. Let $(x,y)\in U$. Then:

$$J_{(x,y)}\varphi = \begin{pmatrix} 1/y & -x/y^2 \\ 1 & 1 \end{pmatrix}.$$

Observe that $\det J_{(x,y)}\varphi = \dfrac{x+y}{y^2} \neq 0$, hence $J_{(x,y)}\varphi$ is invertible and

$$\big(J_{(x,y)}\varphi\big)^{-1} = \frac{y^2}{x+y}\begin{pmatrix} 1 & x/y^2 \\ -1 & 1/y \end{pmatrix},$$

Let $(u,w) = \varphi(x,y)$. Then:

$$J_{(u,w)}\big(\varphi^{-1}\big) = \begin{pmatrix} w/(1+u) - uw/(1+u)^2 & u/(1+u) \\ -w/(1+u)^2 & 1/(1+u) \end{pmatrix} = \begin{pmatrix} y^2/(x+y) & x/(x+y) \\ -y^2/(x+y) & y/(x+y) \end{pmatrix} = (J_{(u,w)}\varphi)^{-1}.$$

3. See Figure 2.

4. a) Since $g = f\circ\varphi$ and since φ is of class C^∞ (hence of class C^1), we deduce that if f is of class C^1 then g is of class C^1. Conversely, since $f = g\circ\varphi^{-1}$, and since φ^{-1} is of class C^∞ (hence of class C^1), we deduce that if g is of class C^1 then f is of class C^1.

   b) Let $(x,y)\in U$. From the expression $g(x,y) = f(x/y, x+y)$ we deduce:

   $$\partial_1 g(x,y) = \frac{1}{y}\partial_1 f(x/y, x+y) + \partial_2 f(x/y, x+y)$$

   and

   $$\partial_2 g(x,y) = -\frac{x}{y^2}\partial_1 f(x/y, x+y) + \partial_2 f(x/y, x+y).$$

   c)

$$\begin{aligned}
\forall (u,w)\in U,\ (u+1)\partial_1 f(u,w) + w\partial_2 f(u,w) = 0 &\iff \forall (u,w)\in U,\ \frac{u+1}{w}\partial_1 f(u,w) + \partial_2 f(u,w) = 0\\
&\iff \forall (u,w)\in U,\ \frac{u+1}{w}\partial_1 f(u,w) + \partial_2 f(u,w) = 0\\
&\iff \forall (x,y)\in U,\ \frac{1}{y}\partial_1 f(x/y, x+y) + \partial_2 f(x/y, x+y) = 0\\
&\iff \forall (x,y)\in U,\ \partial_1 g(x,y) = 0.
\end{aligned}$$

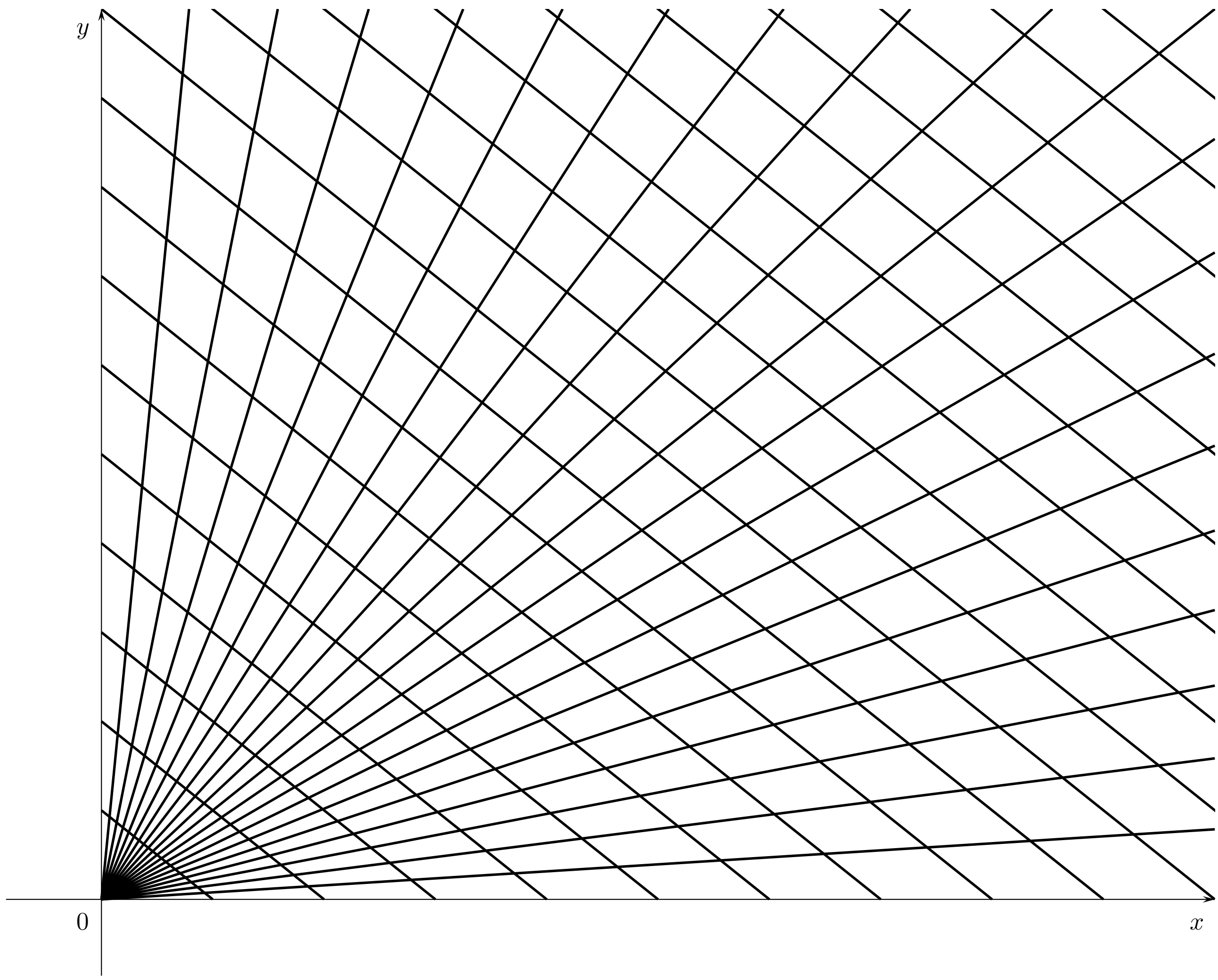

Figure 1. The u- and w-coordinates of Exercise 2: The u-coordinates are the lines passing through the origin, the w-coordinates are the lines with slope -1.

d) This last equation is easy to solve:

$$\begin{cases} g \text{ is of class } C^1 \\ \partial_1 g = 0 \end{cases} \iff \exists A : \mathbb{R}_+^* \to \mathbb{R} \text{ of class } C^1,\ \forall (x,y) \in \mathbb{R}^2,\ g(x,y) = A(y).$$

Hence, the general solution of class C^1 of $(*)$ has the form:

$$\forall (u,w) \in \mathbb{R}^2,\ f(u,w) = A\left(\frac{w}{1+u}\right),$$

where $A : \mathbb{R}_+^* \to \mathbb{R}$ is any function of class C^1.

Exercise 3.

1. Let $f \in E$. Then:

$$\forall t \in [0,1],\ 0 \leq \left|\frac{f(t)}{\sqrt{t}}\right| \leq \frac{\|f\|_\infty}{\sqrt{t}}.$$

By Riemann, we know that the improper integral $\displaystyle\int_0^1 \frac{\mathrm{d}t}{\sqrt{t}}$ is convergent, hence the integral defining $\varphi(f)$ is absolutely convergent, hence convergent.

2. Let $f, g \in E$. Then:

$$\left|\varphi(f) - \varphi(g)\right| = \left|\int_0^1 \frac{f(t)-g(t)}{\sqrt{t}}\,\mathrm{d}t\right|$$

$$\begin{aligned}
&\leq \int_0^1 \frac{\left|f(t)-g(t)\right|}{\sqrt{t}}\,\mathrm{d}t\\
&\leq \int_0^1 \frac{\|f-g\|_\infty}{\sqrt{t}}\,\mathrm{d}t\\
&= \|f-g\|_\infty \int_0^1 \frac{1}{\sqrt{t}}\,\mathrm{d}t\\
&= 2\|f-g\|_\infty.
\end{aligned}$$

3. Since φ is linear, we only need to check that φ is continuous at 0_E. Let $f \in E$. Then, from the previous question,
$$\left|\varphi(f)\right| \leq 2\|f\|_\infty \underset{\|f\|_\infty \to 0}{\longrightarrow} 0,$$
and we conclude that φ is continuous at 0_E.

4. a) Let $n \in \mathbb{N}^*$. Then
$$N(f_n) = \sup_{t\in[0,1]} \left|tf(t)\right| = \sup_{t\in[0,1/n]} \left|-n\sqrt{n}t^2 + \sqrt{n}t\right| = \sqrt{n} \sup_{t\in[0,1/n]} \left|(1-nt)t\right| \leq \sqrt{n}\frac{1}{n} = \frac{1}{\sqrt{n}} \underset{n\to+\infty}{\longrightarrow} 0.$$
Hence $(f_n)_{n\in\mathbb{N}^*}$ converges to 0_E for the norm N.

   b) Let $n \in \mathbb{N}^*$. Then:
$$\begin{aligned}
\varphi(f_n) &= \int_0^{1/n} \frac{-n\sqrt{n}t + \sqrt{n}}{\sqrt{t}}\,\mathrm{d}t\\
&= \int_0^{1/n} -n\sqrt{n}\sqrt{t} + \sqrt{n}\frac{1}{\sqrt{t}}\,\mathrm{d}t\\
&= -n\sqrt{n}\frac{2}{3n^{3/2}} + \frac{2}{\sqrt{n}}\sqrt{n}\\
&= -\frac{2}{3} + 2 = \frac{4}{3} \underset{n\to+\infty}{\longrightarrow} \frac{4}{3} \neq \varphi(0_E) = 0,
\end{aligned}$$
hence φ is not continuous at 0_E.

5. Since φ is continuous from $\left(E, \|\cdot\|_\infty\right)$ but not from (E, N), we conclude that the norms $\|\cdot\|_\infty$ and N are not equivalent.

Exercise 4.

1. Let $(x, y) \in \mathbb{R}^2$. Then:
$$\partial_1 f(x,y) = 1 + yg'(xy), \qquad \partial_2 f(x,y) = xg'(xy).$$

2. a) Since C is the level set of f at level 1, we know that a normal vector to Δ is given by $\overrightarrow{\mathrm{grad}}\, f(1,1) = \left(1+g'(1), g'(1)\right) = (-1,-2)$. Hence an equation of Δ is:
$$\Delta\colon\ -(x-1) - 2(y-1) = 0$$
or, equivalently,
$$\Delta\colon x + 2y = 3.$$

   b) Since $f(1,1) = 1$, we must have $\varphi(1) = 1$. We differentiate the expression in $(*)$ and we obtain:
$$\forall x \in \mathbb{R},\ \partial_1 f\left(x, \varphi(x)\right) + \varphi'(x)\partial_2 f\left(x,\varphi(x)\right) = 0.$$
Hence (evaluating at $x = 1$):
$$\partial_1 f(1,1) + \varphi'(1)\partial_2 f(1,1) = 0,$$
i.e., $-1 - 2\varphi'(1) = 0$, hence $\varphi'(1) = -1/2$. Differentiating again yields:
$$\forall x \in \mathbb{R},\ \partial_{1,1}^2 f\left(x,\varphi(x)\right) + 2\varphi'(x)\partial_{1,2}^2 f\left(x,\varphi(x)\right) + \varphi''(x)\partial_2 f\left(x,\varphi(x)\right) + \varphi'(x)^2\partial_{2,2}^2 f\left(x,\varphi(x)\right) = 0.$$

Hence (evaluating at $x = 1$):

$$\partial^2_{1,1} f(1,1) - \partial^2_{1,2} f(1,1) + \varphi''(1)\partial_2 f(1,1) + \frac{1}{4}\partial^2_{2,2} f(1,1) = 0.$$

We need to determine the second order partial derivatives of f at $(1,1)$: first, for $(x,y) \in \mathbb{R}^2$:

$$\partial^2_{1,1} f(x,y) = y^2 g''(xy), \qquad \partial^2_{1,2} f(x,y) = g'(xy) + xyg''(xy), \qquad \partial^2_{2,2} f(x,y) = x^2 g''(xy),$$

hence

$$\partial^2_{1,1} f(1,1) = 1, \qquad \partial^2_{1,2} f(1,1) = -1, \qquad \partial^2_{2,2} f(1,1) = 1.$$

Hence: $\dfrac{9}{4} - 2\varphi''(1) = 0$, and we conclude that $\varphi''(1) = 9/8 > 0$, hence the graph of φ is above Δ in a neighborhood of $(1,1)$.
See Figure 2. TODO figure.

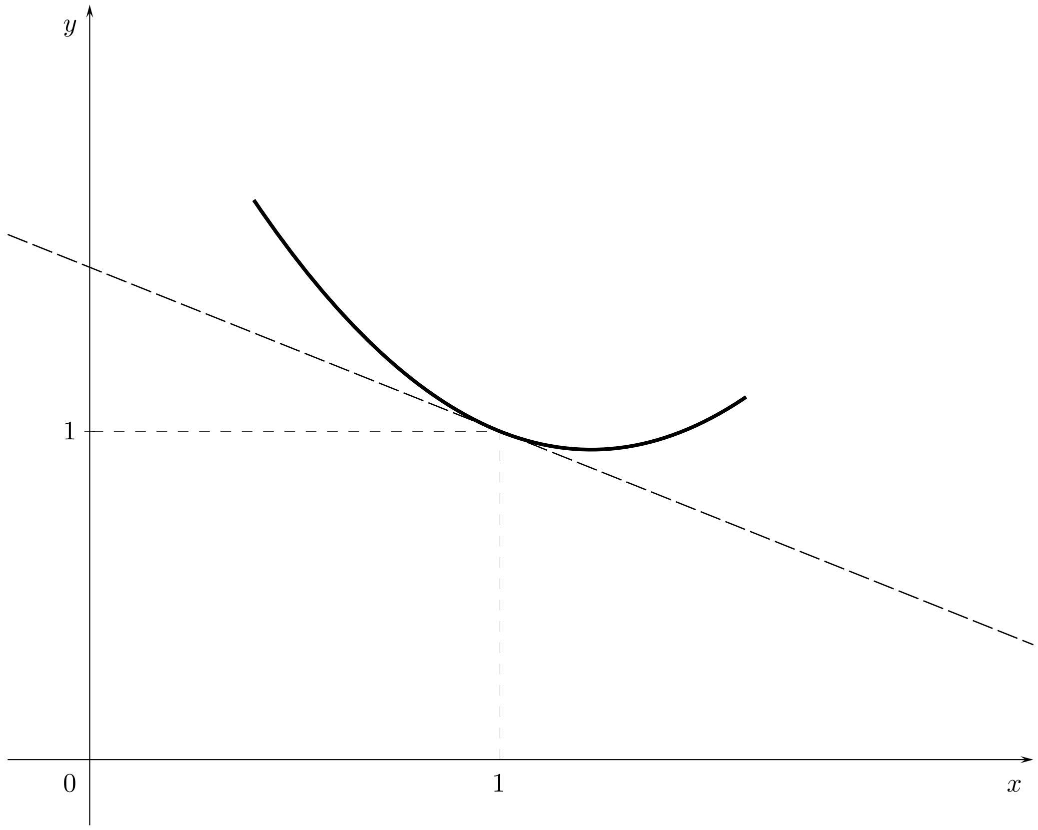

Figure 2. Graph of φ of Exercise 4 in a neighborhood of $(1,1)$, as well as its tangent line Δ at $(1,1)$.