SCAN 2 — S1 — Math Test #2 — 2h

December 9, 2019

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

The marks are given as a guide only; the final marking scheme might differ *slightly* from the marks provided here.

Exercise 1 is (partly) common with EurINSA/ASINSA. Exercise 3 is (partly) common with PCC2.

Exercise 1 (*6 marks*)**.** Let f be the function defined by:

$$f : \begin{array}{rcl} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & \begin{cases} x^2 \ln\bigl(|y|\bigr) + y & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases} \end{array}$$

1. Show that the first order partial derivatives of f at $(0,0)$ exist, and determine them.
2. Is the following proposition true or false (justify your answer)?

   (P) $\qquad \forall u \in \mathbb{R}^2,$ the directional derivative $\nabla_u f(0,0)$ of f at $(0,0)$ in the direction u exists.

3. Show that f is not continuous at $(0,0)$. *Hint. You may conclude after computing the value of the following limit:* $\displaystyle\lim_{t\to 0} f\bigl(t, \mathrm{e}^{-1/t^2}\bigr)$.
4. Is f differentiable at $(0,0)$?

   From now on we define
   $$D = \bigl\{(x,0);\ x \in \mathbb{R}\bigr\} = \mathbb{R} \times \{0\}.$$

5. Let $(x,y) \in \mathbb{R} \setminus D$. Compute the gradient of f at (x,y).
6. Determine the differential of f at $(1,1)$. You may use the notation (e_1', e_2') for the dual basis of $\mathbb{R}^2$.
7. Determine an approximation of $f(1.01\ ,\ 1.02)$.
8. Determine the second order Taylor–Young formula of f at $(1,1)$.

Exercise 2 (*5.5 marks*)**.** Let $U = \mathbb{R}_+^* \times \mathbb{R}_+^*$ We define:

$$\varphi : \begin{array}{rcl} U & \longrightarrow & U \\ (x,y) & \longmapsto & \bigl(x/y, x+y\bigr). \end{array}$$

1. Show that φ is a bijection and determine φ^{-1}. Deduce that φ is a C^∞-diffeomorphism.
2. Let $(x,y) \in U$ and let $(u,w) = \varphi(x,y)$. Check that $\bigl(D_{(x,y)}\varphi\bigr)^{-1} = D_{(u,w)}\bigl(\varphi^{-1}\bigr)$.
3. Plot several coordinates associated with φ, that is, plot in U the curves of the form $x/y = A$ (the u-coordinates) and $x + y = B$ (the w-coordinates) for several values of $A, B \in \mathbb{R}_+^*$.
4. Let $f : U \to \mathbb{R}$ be a differentiable function, and define $g = f \circ \varphi$.

   a) Explain why
   $$f \text{ is of class } C^1 \iff g \text{ is of class } C^1.$$
   b) Determine the first order partial derivatives of g.

   c) Deduce that f is a solution of the following partial differential equation:

   $$(*) \qquad \forall (u,w) \in U, \qquad (u+1)\,\partial_1 f(u,w) \;+\; w\,\partial_2 f(u,w) \;=\; 0$$

   if and only if $\partial_1 g = 0$.

   d) Deduce all the functions $f : U \to \mathbb{R}$ of class C^1 that satisfy Equation $(*)$.

Exercise 3 (*4 marks*). Let $E = C([0,1], \mathbb{R})$ be the real vector space of continuous functions on $[0,1]$ and we define:

$$N : \begin{array}{ccc} E & \longrightarrow & \mathbb{R}_+ \\ f & \longmapsto & \sup\limits_{t \in [0,1]} \left|t f(t)\right|. \end{array}$$

You are given that N is a norm on E. Note that N is only used in Questions 4 and 5 below. We define the following mapping:

$$\varphi : \begin{array}{ccc} E & \longrightarrow & \mathbb{R} \\ f & \longmapsto & \displaystyle\int_0^1 \frac{f(t)}{\sqrt{t}}\, \mathrm{d}t. \end{array}$$

1. Show that φ is well-defined, i.e., that for $f \in E$, the improper integral $\displaystyle\int_0^1 \frac{f(t)}{\sqrt{t}}\, \mathrm{d}t$ is convergent.
2. Show that there exists $k \in \mathbb{R}_+$ such that

$$\forall f, g \in E,\ \left|\varphi(f) - \varphi(g)\right| \leq k \|f - g\|_\infty.$$

3. Deduce that φ is continuous from $(E, \|\cdot\|_\infty)$ to $(\mathbb{R}, |\cdot|)$;
4. We now show that φ is not continuous from (E, N) to $(\mathbb{R}, |\cdot|)$. For this, we define the sequence $(f_n)_{n \in \mathbb{N}^*}$ of elements of E as:

$$\forall n \in \mathbb{N}^*,\ f_n : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \begin{cases} -n\sqrt{n}\, t + \sqrt{n} & \text{if } 0 \leq t \leq 1/n \\ 0 & \text{if } 1/n \leq t \leq 1. \end{cases} \end{array}$$

   You're given that for all $n \in \mathbb{N}^*$, $f_n \in E$, and you don't have to check this fact.

   a) Show that $(f_n)_{n \in \mathbb{N}^*}$ converges to 0_E for the norm N.

   b) Use the sequence $(f_n)_{n \in \mathbb{N}^*}$ to show that φ is not continuous from (E, N) to $(\mathbb{R}, |\cdot|)$.

5. Are the norms N and $\|\cdot\|_\infty$ equivalent?

Exercise 4 (*4.5 marks*). Let $g : \mathbb{R} \to \mathbb{R}$ be a function of class C^2 such that

$$g(1) = 0, \qquad g'(1) = -2, \qquad g''(1) = 1.$$

Define

$$f : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x, y) & \longmapsto & x + g(xy). \end{array}$$

1. Determine the first order partial derivatives of f in terms of g and its derivatives.
2. We define the set C as:

$$C = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, f(x, y) = 1\right\}.$$

   We assume, moreover, that there exists a mapping φ of class C^2 such that

$$\forall (x, y) \in \mathbb{R}^2,\ f(x, y) = 1 \iff y = \varphi(x).$$

   Note that, in particular, we must have:

$$(*) \qquad \forall x \in \mathbb{R},\ f\big(x, \varphi(x)\big) = 1.$$

   a) Determine an equation of the tangent line Δ to C at $(1, 1)$.

   b) Determine the value of $\varphi(1)$, $\varphi'(1)$ and $\varphi''(1)$. *Hint: to determine the value of $\varphi'(1)$ and $\varphi''(1)$ you might want to differentiate the expression in $(*)$.*

   c) Deduce the relative position of C with respect to Δ in a neighborhood of $(1, 1)$, and sketch, on the same figure, the curve C and the line Δ, in a neighborhood of $(1, 1)$.