INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Solution of Math Test #3

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Exercise 1.

1. We apply the Implicit Function Theorem to the function f at A:
   - The function f is of class C^∞ on $\mathbb{R}$,
   - $f(A) = 0$,
   - $\partial_3 f(1, 0, -1) = 3 \neq 0$,

   hence, by the Implicit Function Theorem, there exists an open neighborhood V of $(1, 0)$ in $\mathbb{R}^2$ and an open interval W containing -1, as well as a function $\varphi : V \to W$, of class C^∞ such that:
   $$\forall (x, y) \in V, \forall z \in W,\ \bigl(f(x, y, z) = 0 \iff z = \varphi(y, z)\bigr).$$
   and $\varphi(1, 0) = -1$.

2. It's possible to answer this question by writing the first order Taylor–Young expansion of φ at $(1, 0)$. But here it's probably simpler to just use the gradient of f at A:
   $$\nabla f(A) = (2, 0, 3),$$
   so that an equation of the tangent plane to S at A is:[1]
   $$2(x - 1) + 3(z + 1) = 0.$$

3. By the "moreover" part of the Implicit Function Theorem, we know the form of the first order partial derivatives of φ: for $(x, y) \in V$,
   $$\partial_1\varphi(x, y) = -\frac{\partial_1 f\bigl(x, y, \varphi(x, y)\bigr)}{\partial_3 f\bigl(x, y, \varphi(x, y)\bigr)} = \frac{y^3 - 2x}{-y^2 + 3\varphi(x, y)^2} \qquad \partial_2\varphi(x, y) = -\frac{\partial_2 f\bigl(x, y, \varphi(x, y)\bigr)}{\partial_3 f\bigl(x, y, \varphi(x, y)\bigr)} = \frac{3xy^2 + 2y\varphi(x, y)}{-y^2 + 3\varphi(x, y)^2}$$
   Since φ is of class C^2, we can compute $\partial_2(\partial_1\varphi)(1, 0)$. We then need to differentiate the expression $\partial_1\varphi(x, y)$ with respect to y and evaluate at $(1, 0)$. We're going to decompose the computation as follows: for $(x, y) \in V$, define:
   $$N(x, y) = -y^2 + 2x \qquad \text{and} \qquad D(x, y) = -y^2 + 3\varphi(x, y)^2.$$
   Notice that N and D are the numerator and the denominator of the fraction I wrote for the value of $\partial_1\varphi(x, y)$. Then, $\partial_2 N(1, 0) = 0$ and, noticing that $\partial_2\varphi(1, 0) = 0$ yields:
   $$\partial_2 D(1, 0) = 6\partial_2\varphi(1, 0)\varphi(1, 0) = -6\partial_2\varphi(1, 0) = 0.$$
   So that, according to the quotient rule:
   $$\partial^2_{1,2}\varphi(1, 0) = \frac{\partial_2 N(1, 0)D(1, 0) - N(1, 0)\partial_2 D(1, 0)}{D(1, 0)^2} = 0.$$

4. Let $(x, y, z) \in S$. Then, by definition of S, $f(x, y, z) = 0$, hence, by definition of f, $x^2 - xy^3 - y^2 z + z^3 = 0$, hence (by the standard sign rules):
   $$(-x)^2 - (-x)(-y)^3 - (-y)^2 z + z^3 = 0$$
   hence, by definition of f, $f(-x, -y, z) = 0$, hence, by definition of S, $(-x, -y, z) \in S$.

[1] or, equivalently $2x + 3z = -1$ or, equivalently, $z = -\frac{1}{3}(1 + 2x)$.

Exercise 2.

1. Let $g : \mathbb{R}^2 \to \mathbb{R}$ of class C^2. Then:

$$\begin{cases} g \text{ is of class } C^2 \\ \forall (u,v) \in \mathbb{R}^2,\ \partial^2_{1,2} g(u,v) = u+v \end{cases} \iff \begin{cases} g \text{ is of class } C^2 \\ \forall (u,v) \in \mathbb{R}^2,\ \partial^2_{1,2} g(u,v) = u+v \end{cases}$$

$$\iff \begin{cases} g \text{ is of class } C^2 \\ \exists \varphi : \mathbb{R} \to \mathbb{R} \text{ of class } C^1,\ \forall (u,v) \in \mathbb{R}^2,\ \partial_2 g(u,v) = \frac{u^2}{2} + uv + \varphi(v) \end{cases}$$

$$\iff \begin{cases} g \text{ is of class } C^2 \\ \exists \Phi : \mathbb{R} \to \mathbb{R} \text{ of class } C^2,\ \exists \Psi : \mathbb{R} \to \mathbb{R} \text{ of class } C^1, \\ \qquad \forall (u,v) \in \mathbb{R}^2,\ g(u,v) = \frac{u^2}{2}v + u\frac{v^2}{2} + \Phi(v) + \Psi(u) \end{cases}$$

$$\iff \exists \Phi : \mathbb{R} \to \mathbb{R} \text{ of class } C^2,\ \exists \Psi : \mathbb{R} \to \mathbb{R} \text{ of class } C^2,$$

$$\forall (u,v) \in \mathbb{R}^2,\ g(u,v) = \frac{u^2}{2}v + u\frac{v^2}{2} + \Phi(v) + \Psi(u).$$

2. a) See Figure 3.

Figure 3. Representation of the set U of Exercise 2

b) For $(x,y) \in U$,

$$J_{(x,y)}\varphi = \begin{pmatrix} 0 & \mathrm{e}^y \\ y & x \end{pmatrix}$$

c) Let $(x, y) \in U$ and $(u, v) \in V$. Note that $u > 0$ Then:

$$\varphi(x,y) = (u,v) \iff \begin{cases} u = \mathrm{e}^y \\ v = xy \end{cases} \iff \begin{cases} y = \ln(u) \\ x = \dfrac{v}{\ln(u)}. \end{cases}$$

Moreover, for $(u, v) \in V$, $x \in \mathbb{R}$ and $y < 0$ hence $(x, y) \in U$. Hence, φ is a bijection and its inverse is described by:

$$\begin{aligned} \psi = \varphi^{-1} \ : \quad V &\longrightarrow U \\ (u,v) &\longmapsto \left(\frac{v}{\ln(u)}, \ln(u)\right). \end{aligned}$$

Clearly, φ and φ^{-1} are of class C^∞, hence φ is a C^∞-diffeomorphism.

d) For $(x, y) \in U$,

$$J_{\varphi(x,y)}\left(\varphi^{-1}\right) = \left(J_{(x,y)}\varphi\right)^{-1}.$$

e) See Figure 4.

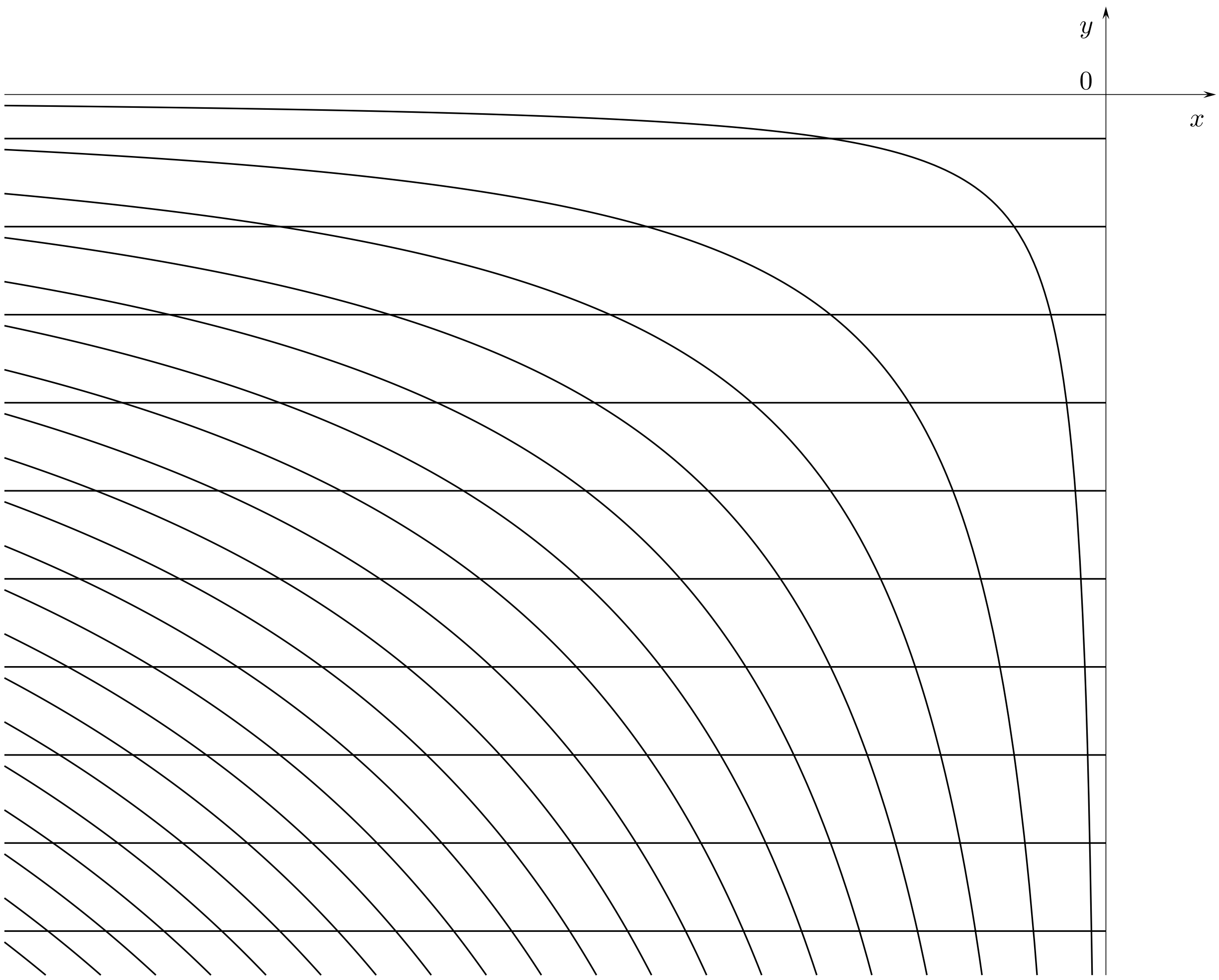

Figure 4. Coordinates associated with the diffeomorphism φ of Exercise 2.

f) Let $f : U \to \mathbb{R}$ of class C^2 and define:

$$\begin{aligned} g \ : \quad V &\longrightarrow \mathbb{R} \\ (u,v) &\longmapsto f\left(\frac{v}{\ln(u)}, \ln(u)\right). \end{aligned}$$

Then, since ψ and f are of class C^2, g is also of class C^2. For $(u, v) \in V$,

$$\partial_2 g(u,v) = \frac{1}{\ln(u)} \partial_1 f\left(\frac{v}{\ln(u)}, \ln(u)\right)$$

and, setting $(x,y)=\psi(u,v)$,

$$\begin{aligned}\partial^2_{1,2}g(u,v)&=-\frac{1}{u\ln(u)^2}\partial_1 f(x,y)-\frac{v}{u\ln(u)^3}\partial^2_{1,1}f(x,y)+\frac{1}{u\ln(u)}\partial^2_{2,1}f(x,y)\\&=-\frac{1}{y^2\mathrm{e}^y}\partial_1 f(x,y)-\frac{x}{y^2\mathrm{e}^y}\partial^2_{1,1}f(x,y)+\frac{1}{y\mathrm{e}^y}\partial^2_{2,1}f(x,y)\\&=-\frac{1}{y^2\mathrm{e}^y}\big(\partial_1 f(x,y)+x\partial^2_{1,1}f(x,y)-y\partial^2_{2,1}f(x,y)\big)\\&=-\frac{1}{y^2\mathrm{e}^y}\big(\partial_1 f(x,y)+x\partial^2_{1,1}f(x,y)-y\partial^2_{2,1}f(x,y)\big)\end{aligned}$$

$$\begin{aligned}-\frac{1}{y^2\mathrm{e}^y}\big(\partial_1 f(x,y)+x\partial^2_{1,1}f(x,y)-y\partial^2_{2,1}f(x,y)+xy^3\mathrm{e}^y+y^2\mathrm{e}^{2y}\big)&=\partial^2_{1,2}g(u,v)-xy-\mathrm{e}^y\\&=\partial^2_{1,2}g(u,v)-u-v.\end{aligned}$$

We hence conclude that:

$$f\text{ is a solution of (E)}\iff\forall(u,v)\in V,\ \partial^2_{1,2}g(u,v)=u+v.$$

g) Hence, from the preliminary question,

$$\begin{aligned}f\text{ is a solution of (E)}&\iff\forall(u,v)\in V,\ \partial^2_{1,2}g(u,v)=u+v\\&\iff\exists\Phi:(1/2,1)\to\mathbb{R}\text{ of class }C^2,\ \exists\Psi:(0,1)\to\mathbb{R}\text{ of class }C^2,\\&\qquad\forall(u,v)\in V,\ g(u,v)=\frac{1}{2}uv(u+v)+\Phi(v)+\Psi(u)\\&\iff\exists\Phi:(1/2,1)\to\mathbb{R}\text{ of class }C^2,\ \exists\Psi:(0,1)\to\mathbb{R}\text{ of class }C^2,\\&\qquad\forall(u,v)\in V,\ g(u,v)=\frac{1}{2}uv(u+v)+\Phi(v)+\Psi(u)\\&\iff\exists\Phi:(1/2,1)\to\mathbb{R}\text{ of class }C^2,\ \exists\Psi:(0,1)\to\mathbb{R}\text{ of class }C^2,\\&\qquad\forall(x,y)\in U,\ f(x,y)=\frac{1}{2}xy\mathrm{e}^y\big(\mathrm{e}^y+xy\big)+\Phi(xy)+\Psi\big(\mathrm{e}^x\big).\end{aligned}$$

Exercise 3.

1. f is clearly of class C^1 on $\mathbb{R}^2\setminus\{(0,0)\}$, and for $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$,

$$\partial_1 f(x,y)=-\frac{2xy^4}{\left(x^2+y^2\right)^2}\qquad\qquad\partial_2 f(x,y)=\frac{2(2x^2+y^2)y^3}{\left(x^2+y^2\right)^2}.$$

Now, the partial derivatives at $(0,0)$: for $t\in\mathbb{R}^*$,

$$\begin{aligned}\frac{f(t,0)-f(0,0)}{t}&=0\underset{t\to0}{\longrightarrow}0\\\frac{f(0,t)-f(0,0)}{t}&=\frac{t^4}{t^2}=t^2\underset{t\to0}{\longrightarrow}0.\end{aligned}$$

Hence the first order partial derivatives of f exist at $(0,0)$ and

$$\partial_1 f(0,0)=\partial_2 f(0,0)=0.$$

Now, for $(x,y)\neq(0,0)$,

$$\begin{aligned}|\partial_1 f(x,y)-\partial_1 f(0,0)|&\leq\frac{2\|(x,y)\|_2^5}{\|(x,y)\|_2^4}=2\|(x,y)\|_2\underset{(x,y)\to(0,0)}{\longrightarrow}0\\|\partial_2 f(x,y)-\partial_2 f(0,0)|&\leq\frac{6\|(x,y)\|_2^5}{\|(x,y)\|_2^4}=6\|(x,y)\|_2\underset{(x,y)\to(0,0)}{\longrightarrow}0.\end{aligned}$$

Hence $\partial_1 f$ and $\partial_2 f$ are continuous at $(0,0)$. Hence $\partial_1 f$ and $\partial_2 f$ are continuous on $\mathbb{R}^2$, and we conclude that f is of class C^1 on $\mathbb{R}^2$.

2. We compute the second order partial derivatives of f at $(1,1)$ (note that f is clearly of class C^2 in a neighborhood of $(1,1)$): for $(x,y) \neq (0,0)$:

$$\begin{aligned}
\partial^2_{1,1} f(x,y) &= \frac{2(3x^2-y^2)y^4}{\left(x^2+y^2\right)^3} \\
\partial^2_{1,2} f(x,y) &= -\frac{8x^3y^3}{\left(x^2+y^2\right)^3} \\
\partial^2_{2,2} f(x,y) &= \frac{2\left(6x^4+3x^2y^2+2y^4\right)y^2}{\left(x^2+y^2\right)^3}
\end{aligned}$$

Hence the Hessian matrix of f at $(1,1)$ is:

$$H_{(1,1)} f = \begin{pmatrix} 1/2 & -1 \\ -1 & 5/2 \end{pmatrix}.$$

We also obtained

$$\partial_1 f(1,1) = -1/2 \qquad\qquad \partial_2 f(1,1) = 3/2$$

and $f(1,1) = 1/2$ so that:

$$f(1+h_x, 1+h_y) \underset{(h_x,h_y)\to(0,0)}{=} \frac{1}{2} - \frac{1}{2}h_x + \frac{3}{2}h_y + \frac{1}{2}\left(\frac{1}{2}h_x^2 - 2h_xh_y + \frac{5}{2}h_y^2\right) + o\left(x^2+y^2\right).$$

3. For $t \in \mathbb{R}^*$:

$$\begin{aligned}
\frac{\partial_1 f(0,t) - \partial_1 f(0,0)}{t} &= 0 \underset{t\to 0}{\longrightarrow} 0 \\
\frac{\partial_2 f(t,0) - \partial_2 f(0,0)}{t} &= 0 \underset{t\to 0}{\longrightarrow} 0
\end{aligned}$$

hence $\partial^2_{1,2} f(0,0) = \partial^2_{2,1} f(0,0) = 0$.

4. Let $x \in \mathbb{R}^*$. Then: $\partial_{1,2} f(x,x) = -1$.

5. Let $y \in \mathbb{R}^*$. Then: $\partial_{1,2} f(0,y) = 0$.

6. We have, on the one hand:

$$\partial_{1,2} f(x,x) \underset{x\to 0}{\longrightarrow} -1$$

and $(x,x) \underset{x\to 0}{\longrightarrow} (0,0)$. Hence, if the limit of $\partial^2_{1,2} f$ existed at $(0,0)$, its value would be -1.

On the other hand:

$$\partial_{1,2} f(0,y) \underset{x\to 0}{\longrightarrow} 0$$

hence, if the limit of $\partial^2_{1,2} f$ existed at $(0,0)$, its value would be 0.

Since $0 \neq -1$, and by uniqueness of the limit of a function at a point, we conclude that $\partial^2_{1,2} f$ is not continuous at $(0,0)$, hence f is not of class C^2.

Exercise 4.

1. Let $n \in \mathbb{N}^*$. Then,

$$0 \leq \frac{1}{3^n n^{1/3}} \leq \frac{1}{3^n}.$$

Now the sequence $(1/3^n)_n$ is a geometric sequence of ratio $1/3 \in (-1,1)$, hence the series $\sum_n 1/3^n$ converges. We conclude, by the comparison test, that (S) converges.

Since

$$\frac{1}{3^n n^{1/3}} \underset{n\to+\infty}{\longrightarrow} 0,$$

and $\mathrm{e}^{1/n} \underset{n\to+\infty}{\longrightarrow} 1$ we have:

$$\ln\left(1 + \frac{1}{3^n n^{1/3}}\right)\mathrm{e}^{1/n} \underset{n\to+\infty}{\sim} \frac{1}{3^n n^{1/3}} > 0$$

hence, by the equivalent test, (S') and (S) have the same nature, hence (S') converges.

2.
$$\mathrm{e}^{1/n^2} - \frac{1}{n^2} \underset{n\to+\infty}{\longrightarrow} 1 \neq 0$$

hence the series (T) diverges.

3. For $n \in \mathbb{N}^*$ define
$$u_n = \left(1 + \frac{\alpha}{n}\right)^{-n^2}.$$

Clearly, for $n \in \mathbb{N}^*$, $u_n \geq 0$, and:
$$u_n^{1/n} = \left(1 + \frac{\alpha}{n}\right)^{-n} = \frac{1}{\left(1 + \frac{\alpha}{n}\right)^n} \underset{n\to+\infty}{\longrightarrow} \mathrm{e}^{-\alpha} < 1 \qquad \textit{since } \alpha > 0.$$

By the root test, we conclude that (R) converges.