INSA | INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

# SCAN 2 — S1 — Math Test #3 — 2h

*January 28, 2019*

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

The marks are given as a guide only; the final marking scheme might differ *slightly* from the marks provided here.

*Exercises 1, 2 and 3 are (partly) common with PCC2.*

**Exercise 1** (*3.5 marks*). Let $f$ be the function defined by:

$$f : \begin{array}{ccc} \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ (x, y, z) & \longmapsto & x^2 - xy^3 - y^2 z + z^3, \end{array}$$

and let $S$ be the surface defined by the equation:

$$S\colon f(x, y, z) = 0,$$

or, more precisely,

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0\}.$$

Let $A = (1, 0, -1) \in \mathbb{R}^3$.

1. Show that, in the neighborhood of $A$, $S$ admits a Cartesian representation of the form $z = \varphi(x, y)$ with $\varphi$ of class $C^\infty$. What is the value of $\varphi(1, 0)$.
2. Determine an equation of the tangent plane to $S$ at $A$.
3. Compute the value of $\partial^2_{1,2}\varphi(1, 0)$.
4. Let $(x, y, z) \in S$. Show that $(-x, -y, z) \in S$. What symmetry property can you deduce about $S$?

**Exercise 2** (*8 marks*).

1. Preliminary question: Find all functions $g : \mathbb{R}^2 \to \mathbb{R}$ of class $C^2$ such that:
$$\forall (u, v) \in \mathbb{R}^2, \qquad \partial^2_{1,2} g(u, v) = u + v.$$

2. Define:
$$U = \left\{(x, y) \in \mathbb{R} \times \mathbb{R}^- \;\middle|\; \frac{1}{2} < xy < 1\right\} \quad \text{and} \quad V = (0, 1) \times \left(\frac{1}{2}, 1\right) = \left\{(u, v) \in \mathbb{R}^2 \;\middle|\; 0 < u < 1,\ \frac{1}{2} < v < 1\right\}.$$
and
$$\varphi : \begin{array}{ccc} U & \longrightarrow & V \\ (x, y) & \longmapsto & (e^y, xy). \end{array}$$
You're given that $\varphi$ is well-defined and you don't need to justify this fact.
   a) Plot $U$.
   b) Compute the Jacobian matrix of $\varphi$.
   c) Show that $\varphi$ is a $C^\infty$-diffeomorphism, and determine $\psi = \varphi^{-1}$
   d) Give a relation between the Jacobian matrix of $\varphi$ and the Jacobian matrix of $\psi$ (specifying the points at which the Jacobian matrices are taken). You are not asked to check that this relation is true.
   e) Plot (you may use the same figure as in Question 2a) several coordinates associated with $\varphi$, that is, curves of the form $e^y = C$ and $xy = C$ for several values of $C$.
   f) For a function $f : U \to \mathbb{R}$ of class $C^2$, we consider the following partial differential equation:
   $$\text{(E)} \qquad \forall (x, y) \in U,\ x\partial^2_{1,1} f(x, y) - y\partial^2_{1,2} f(x, y) + \partial_1 f(x, y) + xy^3 e^y + y^2 e^{2y} = 0.$$
   Define $g = f \circ \psi$, and find a partial differential equation that $g$ satisfies if and only if $f$ satisfies (E).
   g) Find all functions $f : U \to \mathbb{R}$ of class $C^2$ such that $f$ is a solution of Equation (E).

$0V - 2xy^4$

**Exercise 3** (*4.5 marks*). Let

$$f: \begin{array}{rcl} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & \begin{cases} \dfrac{y^4 \; U}{x^2+y^2 \; V} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases} \end{array}$$

$-\dfrac{U'}{U^{m+1}}$

Your are given that $f$ is continuous on $\mathbb{R}^2$.

1. Show that $f$ is of class $C^1$ on $\mathbb{R}^2$, and determine the first-order partial derivatives of $f$ on $\mathbb{R}^2$.
2. Give the second order Taylor–Young expansion of $f$ at $(1,1)$.
3. Compute $\partial^2_{1,2} f(0,0)$ and $\partial^2_{2,1} f(0,0)$.
4. Compute, for $x \in \mathbb{R}^*$, the value of $\partial^2_{1,2}(x,x)$.
5. Compute, for $y \in \mathbb{R}^*$, the value of $\partial^2_{1,2}(0,y)$.
6. Is $f$ of class $C^2$?

**Exercise 4** (*4 marks*).

1. Show that the following numerical series converges:

   (*S*) $$\sum_n \frac{1}{3^n n^{1/3}},$$

   and deduce that the following numerical series converges:

   (*S'*) $$\sum_n \ln\left(1 + \frac{1}{3^n n^{1/3}}\right) e^{1/n}.$$

2. Does the following numerical series converge?

   (*T*) $$\sum_n \left(e^{-1/n^2} - \frac{1}{n^2}\right).$$

3. Find all the $\alpha \in \mathbb{R}_+^*$ such that the following numerical series converges[1]:

   (*R*) $$\sum_n \left(1 + \frac{\alpha}{n}\right)^{-n^2}.$$

   The root test can be useful here.

[1] We recall that for all $x \in \mathbb{R}$,

$$\lim_{n \to +\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$