INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Solution of Math Test #4

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June 12, 2020

Exercise 1.

1.
$$\forall x\in\mathbb{R},\ x\mathrm{e}^{-x}=x\sum_{n=0}^{+\infty}\frac{(-x)^n}{n!}=x\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}x^n=\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}x^{n+1}=\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{(n-1)!}x^n.$$

2. For $x\in(-R,R)$,
$$f'(x)=\sum_{n=1}^{+\infty}a_nnx^{n-1},$$
$$xf'(x)=\sum_{n=0}^{+\infty}a_nnx^n,$$
$$f''(x)=\sum_{n=2}^{+\infty}a_nn(n-1)x^{n-2}$$
$$xf''(x)=\sum_{n=2}^{+\infty}a_nn(n-1)x^{n-1}=\sum_{n=1}^{+\infty}a_{n+1}(n+1)nx^n,$$
hence
$$xf''(x)+xf'(x)+f(x)=a_0+\sum_{n=1}^{+\infty}\big(a_{n+1}(n+1)n+na_n+a_n\big)x^n=a_0+\sum_{n=1}^{+\infty}(n+1)(a_{n+1}n+a_n)x^n$$

3. By the identity theorem, f is a solution if and only if:
$$a_0=0,\ \text{and},\ \forall n\in\mathbb{N}^*,\ a_{n+1}=-\frac{a_n}{n}.$$
It is easy to find a closed form form $(a_n)_{n\in\mathbb{N}}$:
$$\begin{cases}a_0=0\\ \forall n\in\mathbb{N}^*,\ a_n=\frac{(-1)^{n-1}}{(n-1)!}a_1.\end{cases}$$
and we recognize a_1g. Hence the general solution of this differential equation that possesses a power series expansion is Kg for $K\in\mathbb{R}$, and its radius of convergence is $+\infty$

Exercise 2.

1. Set $v_0=e_0$. Notice that $\|e_0\|_2=1$. and $v_1=e_1+\lambda e_0$ with
$$\lambda=-\frac{\langle e_1\,|\,e_0\rangle}{\langle e_0\,|\,e_0\rangle}=-\frac{1}{2},$$
hence $v_1=e_1-\frac{1}{2}e_0$. Now
$$\|v_1\|^2=\langle v_1\,|\,v_1\rangle=\int_0^1\left(t-\frac{1}{2}\right)^2\,\mathrm{d}t=\frac{1}{12}.$$
We hence set:
$$w_0=e_0,\ w_1=\sqrt{12}\left(e_1-\frac{1}{2}e_0\right)=\sqrt{3}(2e_1-e_0).$$

2. Since (w_0,w_1) is an orthonormal basis of F:
$$p_F(e_2)=\langle e_2\,|\,w_0\rangle w_0+\langle e_2\,|\,w_1\rangle w_1.$$

Now,

$$\alpha = \langle e_2 \,|\, w_0 \rangle = \frac{1}{3}$$

and

$$\beta = \langle e_2 \,|\, w_1 \rangle = \sqrt{3}\big(2\langle e_2 \,|\, e_1 \rangle - \langle e_2 \,|\, e_0 \rangle\big) = \sqrt{3}\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\sqrt{3}}{6},$$

hence

$$p_F(e_2) = \frac{1}{3} w_0 + \frac{\sqrt{3}}{6} w_1.$$

3. We notice that

$$m = \inf_{u \in F} \|e_2 - u\|^2,$$

and by the optimization property of the orthogonal projection,

$$m = \big\|e_2 - p_F(e_2)\big\|^2.$$

Moreover, we know that $p_F(e_2) \perp (e_2 - p_F(e_2))$, hence, by the Pythagorean Theorem,

$$\|e_2\|^2 = \big\|p_F(e_2) + (e_2 - p_F(e_2))\big\|^2 = \big\|p_F(e_2)\big\|^2 + \|e_2 - p_F(e_2)\|^2$$

so that

$$m = \|e_2\|^2 - \big\|p_F(e_2)\big\|^2 = \frac{1}{5} - (\alpha^2 + \beta^2)$$

where we used $\big\|p_F(e_2)\big\|^2 = \alpha^2 + \beta^2$ since (w_0, w_1) is an orthonormal basis of F. Hence

$$m = \frac{1}{5} - \left(\frac{1}{9} + \frac{1}{12}\right) = \frac{1}{180}.$$