INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — S1 — Math Test #4 — 45 minutes

June 12, 2020

No documents, no calculators, no cell phones or electronic devices allowed.

All your answers must be fully (but concisely) justified, unless noted otherwise.

The marks are given as a guide only; the final marking scheme might differ *slightly* from the marks provided here.

J'atteste sur l'honneur que je ne tricherai pas, notamment en respectant les consignes données dans le sujet et en n'ayant aucun échange avec une tierce personne autre que l'enseignant(e) qui accompagne l'évaluation. Le rendu d'une copie à cette épreuve vaut acceptation de cet engagement.

Exercise 1 (*10 marks*).

1. Give the power series expansion of the following expression, as well as its radius of convergence:

$$g(x) = x\mathrm{e}^{-x}.$$

2. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and let f be the function defined by the associated power series, assumed to be of radius $R > 0$:

$$\forall x \in (-R, R),\ f(x) = \sum_{n=0}^{+\infty} a_n x^n,$$

   Write the following expression, defined for $x \in (-R, R)$, as the sum of a power series:

$$xf''(x) + xf'(x) + f(x).$$

3. Find the general solution of the following differential equation, that possesses a power series expansion:

$$xy''(x) + xy'(x) + y(x) = 0,$$

   as well as the radius of convergence of the power series.

Exercise 2 (*10 marks*). Let $E = C\big([0,1], \mathbb{R}\big)$ be the vector space of real-valued continuous functions on $[0,1]$, together with the inner product $\langle\cdot \mid \cdot\rangle$ defined by:

$$\forall f, g \in E,\ \langle f \mid g\rangle = \int_0^1 f(t)g(t)\,\mathrm{d}t.$$

Let e_0 and e_1 be the vectors of E defined by:

$$e_0 : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{R} \\ t & \longmapsto & 1 \end{array} \qquad e_1 : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{R} \\ t & \longmapsto & t \end{array} \qquad \text{and} \qquad e_2 : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{R} \\ t & \longmapsto & t^2 \end{array}$$

and let $F = \operatorname{Span}\{e_0, e_1\}$. You're given that $\mathscr{B} = (e_0, e_1)$ is a basis of F.

1. Determine an orthonormal basis $\mathscr{B}' = (w_0, w_1)$ of F.
2. Determine the orthogonal projection $p_F(e_2)$ of e_2 on F, and give its coordinates $\big[p_F(e_2)\big]_{\mathscr{B}'} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ in the basis $\mathscr{B}'$.
3. Define:

$$m = \inf_{a,b\in\mathbb{R}} \int_0^1 \big(t^2 - a - bt\big)^2\,\mathrm{d}t.$$

   Show, using orthogonal projections, that m exists and

$$m = \frac{1}{5} - \big(\alpha^2 + \beta^2\big),$$

   and determine the value of m.