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INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Quiz #2 — 12'

October 3, 2019

Name: Gb Savin

Exercise 1. Let $(E, \|\cdot\|)$ be a normed vector space and let U be a subset of E. In the following you will find definitions for $\bar{U}$ and $\mathring{U}$ as well as definitions that correspond to neither. Fill in the blank with "$\bar{U}$," "$\mathring{U}$" and "neither".

$\{u \in E \mid \forall r > 0,\ \mathring{B}(u,r) \subset U\} =$ neither

$\{u \in E \mid \exists r > 0,\ \mathring{B}(u,r) \subset U\} =$ $\mathring{U}$

$\{u \in E \mid \forall r > 0,\ \mathring{B}(u,r) \cap U \neq \emptyset\} =$ $\bar{U}$

$\{u \in E \mid \exists r > 0,\ \mathring{B}(u,r) \cap U \neq \emptyset\} =$ neither

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Exercise 2. Let $\alpha \in \mathbf{R}$, and define the following improper integral:

$$I_\alpha = \int_0^{+\infty} \frac{\arctan(t)}{t^\alpha}\, dt.$$

Fill in the blank (no justifications required):

I_α converges $\iff$ $\alpha \in (1, 2)$

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Exercise 3. Let $E = C([0,1])$ be the vector space of real-valued continuous functions on $[0,1]$. For $f \in E$ we define

$$\|f\|_1 = \int_0^1 |f(t)|\, dt.$$

You're given that $\|\cdot\|_1$ is a norm on E (you all recognized the 1-norm on E) and you don't have to justify this fact. We define the following elements of E:

$$\ell : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbf{R} \\ t & \longmapsto & 1, \end{array} \qquad \forall n \in \mathbf{N}^*,\ u_n : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbf{R} \\ t & \longmapsto & t^{1/n} \end{array}$$

a) Let $n \in \mathbf{N}^*$. Compute the distance d_n (with respect to the norm $\|\cdot\|_1$) between u_n and ℓ. No justifications required, only give your final answer.

$d_n = 1 - \dfrac{1}{\frac{1}{n} + 1}$

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b) Does the sequence $(u_n)_{n \geq 1}$ converge in $(E, \|\cdot\|)$? Justify your answer (as concisely as possible).

We have $d_n = \|u_n - \ell\|_1 \xrightarrow[n \to \infty]{} 1 - 1 = 0$ hence according to the definition of the convergence of a sequence of vectors, $(u_n)_{n \geq 1}$ converges to ℓ in $(E, \| \|)$

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INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Quiz #1 — 12'

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September 26, 2019

Name: Cote Smrim

Exercise 1. You're given that the following improper integral is convergent:

$$I = \int_0^{+\infty} t^2 e^{-t^3}\,dt.$$

Determine the value of I (no justifications required).

$I = \int_{-\infty}^{0} \frac{e^u}{3} du = \frac{1}{3}$ ✓

Exercise 2. Let $E = \mathbf{R}^4$, and define the following vectors of E:

$$u = (-1, 4, 2, 3), \qquad v = (1, 1, -2, 1).$$

Determine the distance d between u and v with respect to the norm $\|\cdot\|_1$.

$d = \|u - v\|_1 = \|(-2, 3, 4, 2)\|_1 = 2 + 3 + 4 + 2 = 11$ ✓

Exercise 3. Let E be a vector space over $\mathbf{R}$. Recall the definition of "N is a norm on E."

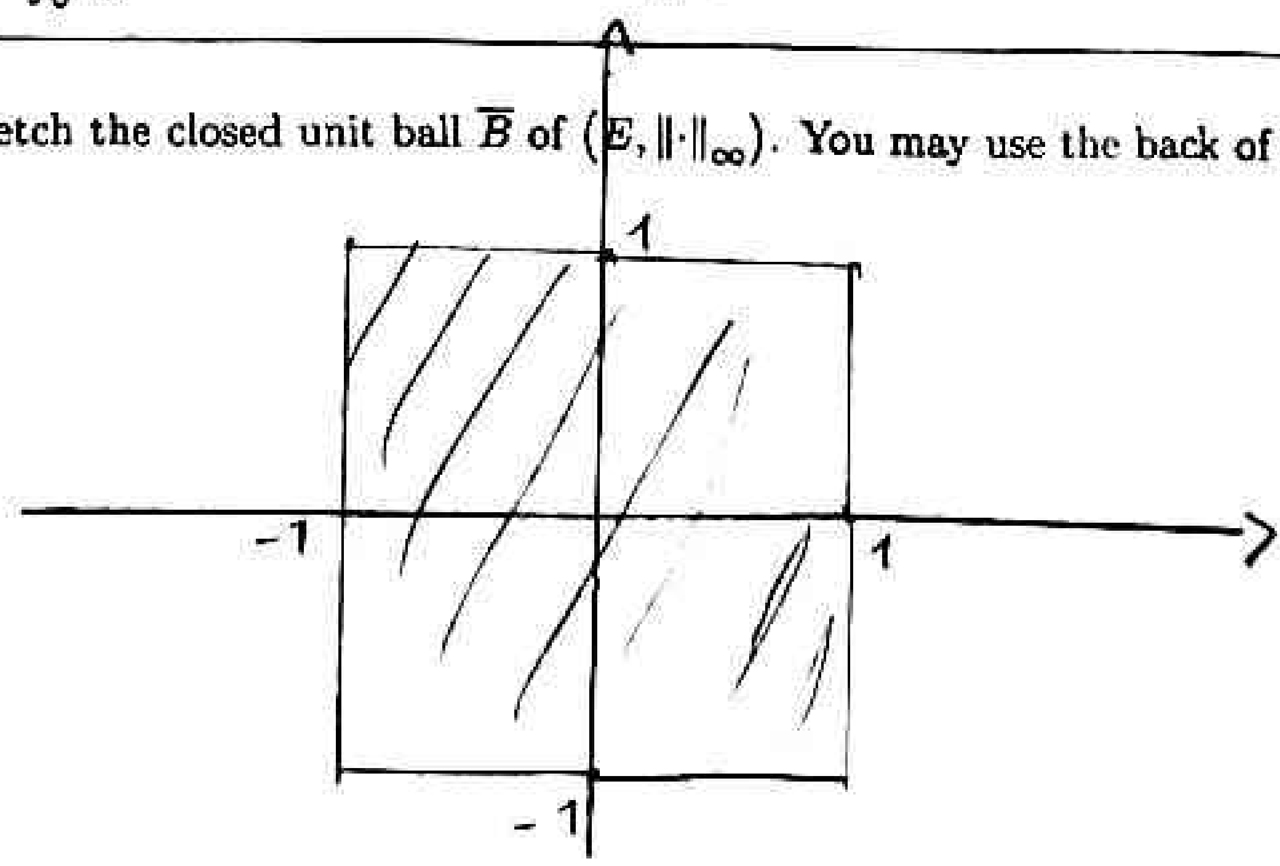
$N : E \longrightarrow \mathbb{R}^+$ (N must be a mapping from E to $\mathbb{R}^+$) ✓

- $\forall u \in E$ $\boxed{N(u) = 0 \Rightarrow u = 0}$ ✓
- $\forall u \in E, \forall \lambda \in \mathbb{R}, N(\lambda u) = |\lambda| N(u)$ ✓
- $\forall (u, v) \in E^2, N(u+v) \leq N(u) + N(v)$ ✓

Exercise 4. Let $\alpha \in \mathbf{R}$. Fill in the blanks:

- the improper integral $\int_1^{+\infty} \frac{dt}{t^\alpha}$ converges $\iff$ $\alpha > 1$ ✓
- the improper integral $\int_0^1 \frac{dt}{t^\alpha}$ diverges $\iff$ $\alpha \geqslant 1$

Exercise 5. Let $E = \mathbf{R}^2$. Sketch the closed unit ball $\overline{B}$ of $(E, \|\cdot\|_\infty)$. You may use the back of this sheet if you don't have enough room below.



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