SCAN 2 — Solution of Math Test #1

Romaric Pujol, romaric.pujol@insa-lyon.fr

October 19, 2020

Exercise 1.

1. The function $t \mapsto t^5 e^{-t^2}$ is continuous on $[0,+\infty)$ hence I_1 is improper at $+\infty$ only. Now,

$$t^7 e^{-t^2} \underset{t\to+\infty}{\longrightarrow} 0$$

hence there exists $A>0$ such that

$$\forall t\in[A,+\infty),\ 0\leq t^7 e^{-t^2}\leq 1,$$

i.e.,

$$\forall t\in[A,+\infty),\ 0\leq t^5 e^{-t^2}\leq \frac{1}{t^2},$$

and we conclude, by the Comparison Test, that I_1 converges.

2. The function $t\mapsto \ln\left(1+\frac{1}{t^2}\right)$ is continuous on $[1,+\infty)$, hence I_2 is improper at $+\infty$ only. Let $A>1$. By an integration by parts (differentiating the ln and antidifferentiating 1):

$$\begin{aligned}\int_1^A \ln\left(1+\frac{1}{t^2}\right)\mathrm{d}t &= \left[t\ln\left(2+\frac{1}{t^2}\right)\right]_{t=1}^{t=A} - \int_1^A t\frac{-2/t^3}{1+1/t^2}\,\mathrm{d}t\\ &= A\ln\left(1+\frac{1}{A^2}\right)-\ln(2)+2\int_1^A\frac{1}{1+t^2}\,\mathrm{d}t\\ &= A\ln\left(1+\frac{1}{A^2}\right)-\ln(2)+2\arctan(A)-2\arctan(1)\\ &= A\ln\left(1+\frac{1}{A^2}\right)-\ln(2)+2\arctan(A)-\frac{\pi}{2}\\ &= -\ln(2)+\frac{\pi}{2}\end{aligned}$$

Exercise 2.

1. The function $x\mapsto \frac{1}{\sqrt{x}+x^{3/2}}$ is continuous on $(0,+\infty)$ hence I is improper at 0^+ and at $+\infty$. Now,

$$\frac{1}{\sqrt{x}+x^{3/2}} \underset{x\to0^+}{\sim} \frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}} > 0,$$

hence by the Equivalent Test, I converges at 0^+, and

$$\frac{1}{\sqrt{x}+x^{3/2}} \underset{x\to+\infty}{\sim} \frac{1}{x^{3/2}} > 0,$$

hence by the Equivalent Test, I converges at $+\infty$. Hence I converges.

2. Let $A,B\in\mathbb{R}$ such that $0<A<B$. We use the substitution $u=\sqrt{x}$, so that $\mathrm{d}x=2u\,\mathrm{d}u$:

$$\int_A^B\frac{\mathrm{d}x}{\sqrt{x}+x^{3/2}} = \int_{\sqrt{A}}^{\sqrt{B}}\frac{2u\,\mathrm{d}u}{u+u^3} = \int_{\sqrt{A}}^{\sqrt{B}}\frac{2\,\mathrm{d}u}{1+u^2} = 2\arctan\sqrt{B}-2\arctan\sqrt{A} \underset{A\to0^+,\ B\to+\infty}{\longrightarrow}\pi.$$

Hence $I=\pi$.

3.
   - If $\alpha<1/2$ then

$$\frac{1}{\sqrt{x}+x^{\alpha}} \underset{x\to+\infty}{\sim} \frac{1}{x^{1/2}} > 0$$

hence, by the Equivalent Test (and Riemann at $+\infty$), I_α diverges.

- If $\alpha = 1/2$ then

$$\frac{1}{\sqrt{x}+x^{\alpha}} = \frac{1}{2x^{1/2}}$$

and by Riemann at $+\infty$, I_α diverges too.

- If $\alpha > 1/2$ then

$$\frac{1}{\sqrt{x}+x^{\alpha}} \underset{x\to 0^+}{\sim} \frac{1}{\sqrt{x}} > 0,$$

hence by the Equivalent Test, I_α converges at 0^+ and

$$\frac{1}{\sqrt{x}+x^{\alpha}} \underset{x\to +\infty}{\sim} \frac{1}{x^{\alpha}} > 0,$$

and we conclude, by the Equivalent Test, that I_α converges at $+\infty$ if and only if $\alpha > 1$.

Conclusion: I_α converges if and only if $\alpha > 1$.

Exercise 3.

1. The largest interval with endpoint $+\infty$ where the function $t \mapsto \ln\left(\cos\left(\frac{1}{t}\right)\right)$ is continuous is $(2/pi, +\infty)$. Hence:

$$I_a \text{ is improper at } +\infty \text{ only} \iff a > \frac{2}{\pi}.$$

2. Since $\cos(1/t) \underset{t\to+\infty}{\longrightarrow} 1$ we obtain, by the well-known equivalent $\ln(X) \underset{X\to 1}{\sim} X - 1$:

$$\ln\left(\cos\left(\frac{1}{t}\right)\right) \underset{t\to+\infty}{\sim} \cos\left(\frac{1}{t}\right) - 1 \underset{t\to+\infty}{\sim} -\frac{1}{2t^2} < 0.$$

Hence by the Equivalent Test (and Riemann at $+\infty$), I_α converges.

Exercise 4.

1. Let $n \geq 1$. The function $t \mapsto \frac{1}{t^n(1+t)}$ is continuous on $[1, +\infty)$, hence I_n is only improper at $+\infty$. Now,

$$\frac{1}{t^n(1+t)} \underset{t\to+\infty}{\sim} \frac{1}{t^{n+1}} > 0.$$

By the Equivalent Test (and Riemann at $+\infty$), I_n converges (since $n+1 > 1$ since $n \geq 1 > 0$).
To compute the value of I_1 we use the following partial fraction decomposition:

$$\frac{1}{T(1+T)} = \frac{1}{T} - \frac{1}{1+T}$$

Let $X > 1$. Then

$$\int_1^X \frac{\mathrm{d}t}{t(1+t)} = \int_1^X \frac{\mathrm{d}t}{t} - \int_1^X \frac{\mathrm{d}t}{1+t} = \ln(X) - \ln(X+1) + \ln(2) = \ln\left(\frac{X}{X+1}\right) + \ln(2) \underset{X\to+\infty}{\longrightarrow} \ln(2).$$

Hence $I_1 = \ln(2)$.

2. Let $t \in [1, +\infty)$. Then:

$$2 \leq 1 + t \leq t + t = 2t$$

hence

$$2t^n \leq t^n(1+t) \leq t + t = 2t^{n+1}$$

hence

$$\frac{1}{2t^{n+1}} \leq \frac{1}{t^n(1+t)} \leq \frac{1}{2t^n}.$$

Now for $\alpha > 1$,

$$\int_1^{+\infty} \frac{\mathrm{d}t}{t^{\alpha}} = \frac{1}{a-1},$$

so that (since $n \geq 2 > 1$):

$$\frac{1}{2n} = \int_1^{+\infty} \frac{\mathrm{d}t}{2t^{n+1}} \leq I_n = \int_1^{+\infty} \frac{\mathrm{d}t}{t^n(1+t)} \leq \int_1^{+\infty} \frac{\mathrm{d}t}{2t^n} = \frac{1}{2(n-1)}.$$

3. a) Let $n \geq 1$ and $t \in [1, +\infty)$. Then

$$\frac{1}{t^n(1+t)} + \frac{1}{t^{n+1}(1+t)} = \frac{t+1}{t^{n+1}(1+t)} = \frac{1}{t^{n+1}}.$$

Hence:

$$I_n + I_{n+1} = \int_1^{+\infty} \frac{\mathrm{d}t}{t^{n+1}} = \frac{1}{n}.$$

b) Let $n \geq 2$. Then:

$$\begin{aligned}
\sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k} &= \sum_{k=1}^{n-1} (-1)^{k-1}(I_k + I_{k+1}) \\
&= \sum_{k=1}^{n-1} (-1)^{k-1} I_k + \sum_{k=1}^{n-1} (-1)^{k-1} I_{k+1} \\
&= \sum_{k=1}^{n-1} (-1)^{k-1} I_k + \sum_{k=2}^{n} (-1)^{k} I_k && \textit{shift of index in the second sum} \\
&= \sum_{k=1}^{n-1} (-1)^{k-1} I_k - \sum_{k=2}^{n} (-1)^{k-1} I_k \\
&= I_1 - (-1)^{n-1} I_n,
\end{aligned}$$

hence the result.

4. From Question 2 and the Squeeze Theorem we conclude that $I_n \underset{n \to +\infty}{\longrightarrow} 0$, hence

$$\ell = I_1 = \ln(2).$$

Exercise 5. Let

$$\begin{array}{rccc} \varphi : & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (x, y) & \longmapsto & (2x + y, x + y). \end{array}$$

Clearly φ is linear and

$$\forall u \in \mathbb{R}^2,\ N(u) = \left\|\varphi(u)\right\|_\infty.$$

This already shows that N satisfies the triangle inequality and the positive homogeneity (and that N takes values in $\mathbb{R}_+$). In order to show that N also satisfy the separation property, we only need to show that φ is injective: the matrix of φ in the standard basis of $\mathbb{R}^2$ is:

$$[\varphi]_{\text{std}} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

the determinant of which is $\det \varphi = 1 \neq 0$, hence φ is a bijection.
We know that the closed ball $\overline{B}_N$ associated with N is obtained as:

$$\varphi^{-1}\left(\overline{B}_\infty\right)$$

where $\overline{B}_\infty$ is the unit ball associated with the ∞-norm.
We now compute the matrix of φ^{-1}:

$$[\varphi^{-1}]_{\text{std}} = [\varphi]_{\text{std}}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

The images of (two of) the vertices of $\overline{B}_\infty$ by φ^{-1} are hence:

$$\varphi^{-1}(1, 1) = (0, 1), \qquad \varphi^{-1}(1, -1) = (-2, -3).$$

From this we can deduce $\overline{B}_\infty$ (see Figure 1).

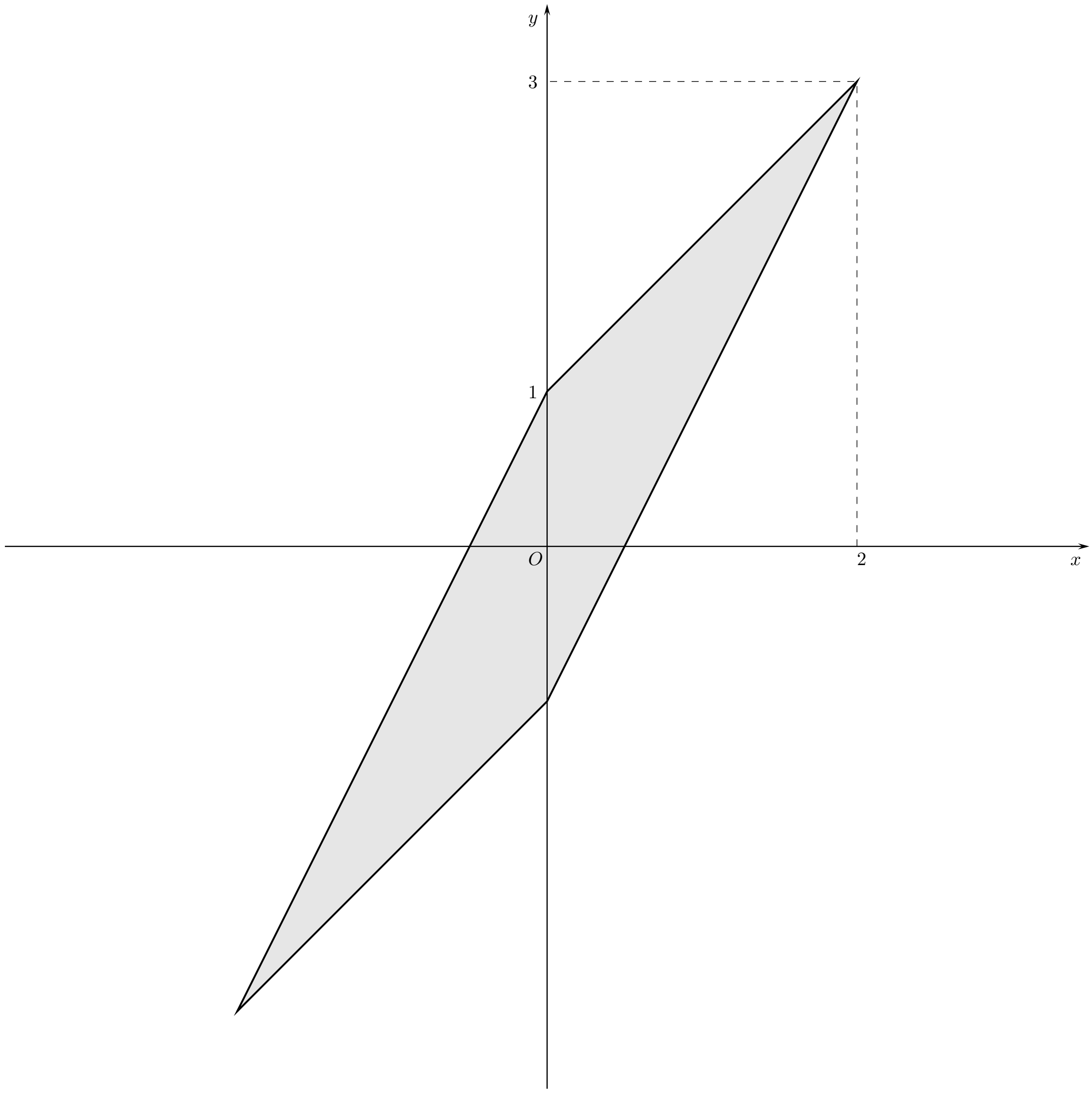

Figure 1. Closed unit ball for the norm N of Exercise 5

Exercise 6.

1. Let $n \in \mathbb{N}$. Then:
$$\|u_n - 0_E\|_1 = \|u_n\|_1 = \int_0^1 \left|u_n(t)\right| \mathrm{d}t = \int_0^1 \sqrt{n}t^n \,\mathrm{d}t = \frac{\sqrt{n}}{n+1}$$
hence $\|u_n - 0_E\|_1 \underset{n\to+\infty}{\longrightarrow} 0$, hence $(u_n)_{n\in\mathbb{N}}$ converges to 0_E for the 1-norm.

2. Let $n \in \mathbb{N}$. Then:
$$\|u_n - 0_E\|_2 = \|u_n\|_2 = \int_0^1 u_n(t)^2 \,\mathrm{d}t = \int_0^1 nt^{2n} \,\mathrm{d}t = \frac{n}{2n+1},$$
hence $\|u_n - 0_E\|_2 \underset{n\to+\infty}{\not\longrightarrow} 0$, hence $(u_n)_{n\in\mathbb{N}}$ doesn't converge to 0_E for the 2-norm.