INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — S1 — Math Test #1 — 2h

October 19, 2020

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

The marks are given as a guide only; the final marking scheme might differ *slightly* from the marks provided here.

Exercises 1, 2, 3 and 4 are common with PCC2.

Exercise 1 (*3 points*)**.** The two questions of this exercise are independent of each other.

1. Study the convergence of the improper integral

$$I_1 = \int_0^{+\infty} t^5 \mathrm{e}^{-t^2}\,\mathrm{d}t.$$

2. Use an integration by parts to compute the value of the following integral:

$$I_2 = \int_1^{+\infty} \ln\left(1 + \frac{1}{t^2}\right)\mathrm{d}t.$$

Exercise 2 (*5 points*)**.** Let I be the following improper integral:

$$I = \int_0^{+\infty} \frac{\mathrm{d}x}{\sqrt{x} + x^{3/2}}.$$

1. Justify that I converges.
2. Use a substitution to show that $I = \pi$.
3. For what values of $\alpha \in \mathbb{R}$ is the improper integral

$$I_\alpha = \int_0^{+\infty} \frac{\mathrm{d}x}{\sqrt{x} + x^{a}}$$

convergent? (You may distinguish the cases where α is smaller or greater than $1/2$).

Exercise 3 (*2.5 points*)**.** Let $a \in \mathbb{R}$ and define

$$I_a = \int_a^{+\infty} \ln\left(\cos\left(\frac{1}{t}\right)\right)\mathrm{d}t.$$

1. For what values of a is the improper integral I_a only improper at $+\infty$?
2. For these values of a, is the integral I_a convergent?

Exercise 4 (*6.5 points*)**.** The goal of this exercise is to compute the value of the following limit:

$$\ell = \lim_{n\to+\infty} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}.$$

For $n \geq 1$ we set

$$I_n = \int_1^{+\infty} \frac{\mathrm{d}t}{t^n(1+t)}.$$

1. Justify that for $n \geq 1$, I_n is convergent. Compute the value of I_1.
2. Show that

$$\forall n \geq 2,\ \frac{1}{2n} \leq I_n \leq \frac{1}{2(n-1)}.$$

   You may use appropriate inequalities for the expression $\dfrac{1}{t^n(1+t)}$ for $t \in [1, +\infty)$.

3. a) Show that

$$\forall n \geq 1,\ I_n + I_{n+1} = \frac{1}{n}.$$

   b) Deduce that

$$\forall n \geq 2,\ (-1)^n I_n = \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k} - I_1.$$

4. Deduce the value of ℓ.

Exercise 5 (*1.5 point*)**.** Let $E = \mathbb{R}^2$ and let

$$N : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & \max\{|2x+y|, |x+y|\}. \end{array}$$

Show, as concisely as possible, that N is a norm on E, and plot its unit (closed) ball.

Exercise 6 (*1.5 points*)**.** Let $E = C([0,1])$ be the vector space of (real valued) continuous functions on $[0,1]$. Let $\alpha \in \mathbb{R}$. We define the sequence $(u_n)_{n\in\mathbb{N}}$ of elements of E as follows:

$$\forall n \in \mathbb{N},\ u_n : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \sqrt{n}\, t^n. \end{array}$$

1. Do we have $\lim\limits_{n\to+\infty} u_n = 0_E$ for the 1-norm $\|\cdot\|_1$?
2. Do we have $\lim\limits_{n\to+\infty} u_n = 0_E$ for the 2-norm $\|\cdot\|_2$?

Justify your answers.