INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Solution of Math Test #2

Romaric Pujol, romaric.pujol@insa-lyon.fr

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Exercise 1.

- Let $t \in \mathbb{R}^*$. Then:
$$\frac{f(t,0)-f(0,0)}{t} = \frac{\dfrac{t^3+0}{t^2+0}-0}{t} = 1 \xrightarrow[t\to 0]{} 1,$$
hence $\partial_1 f(0,0)$ exists and its value is 1.
- Let $t \in \mathbb{R}^*$. Then:
$$\frac{f(0,t)-f(0,0)}{t} = \frac{\dfrac{0+t}{0+t^2}-0}{t} = \frac{1}{t^3},$$
the limit of which as $t \to 0$ doesn't exist. Hence $\partial_2 f(0,0)$ doesn't exist.

Since $\partial_2 f(0,0)$ doesn't exist, we can conclude that f is not differentiable at $(0,0)$.

Exercise 2. Since $U = \mathbb{R}^2 \setminus \{(0,0)\}$ is an open set and the expression defining f there is continuous, f is continuous on U. We now address the continuity of f at $(0,0)$: let $(x,y) \in U$ (notice that U is a punctured neighborhood of $(0,0)$). Then:

$$\left|f(x)\right| = \left|\frac{x^2y+xy^2}{x^2+2y^2}\right| \leq \frac{|x|^2|y|+|x||y|^2}{2x^2+2y^2} \leq \frac{2\left\|(x,y)\right\|_2^3}{2\left\|(x,y)\right\|_2^2} = \left\|(x,y)\right\|_2 \xrightarrow[\|(x,y)\|_2\to 0]{} 0,$$

hence f is also continuous at $(0,0)$.

Exercise 3.

$$\partial_1 g(x,y) = u\big(xy, 2x+f(y)\big) + xy\partial_1 u\big(xy,2x+f(y)\big) + 2x\partial_2 u\big(xy,2x+f(y)\big)$$
$$\partial_2 g(x,y) = x^2\partial_1 u\big(xy,2x+f(y)\big) + xf'(y)\,\partial_2 u\big(xy,2x+f(y)\big)$$

Exercise 4.

1.
$$J_{(u,v)}\varphi = \begin{pmatrix} 2u & 2v \\ v & u \end{pmatrix}.$$

2.
$$J_{(u,v)}F = J_{\varphi(u,v)}f\ J_{(u,v)}\varphi.$$

3. Hence:
$$\begin{pmatrix}\partial_1 F(u,v) & \partial_2 F(u,v)\end{pmatrix} = \begin{pmatrix}\partial_1 f\big(\varphi(u,v)\big) & \partial_2 f\big(\varphi(u,v)\big)\end{pmatrix}\begin{pmatrix} 2u & 2v \\ v & u \end{pmatrix},$$
from which we deduce:
$$\partial_1 F(u,v) = 2u\partial_1 f\big(u^2+v^2,uv\big) + v\partial_2 f\big(u^2+v^2,uv\big),$$
$$\partial_2 F(u,v) = 2v\partial_1 f\big(u^2+v^2,uv\big) + u\partial_2 f\big(u^2+v^2,uv\big).$$

4. We first determine F: for $(u,v) \in \mathbb{R}^2$,
$$F(u,v) = \big(u^2+v^2\big)^2 - 4u^2v^2 = u^4 - 2u^2v^2 + v^4 = \big(u^2-v^2\big)^2.$$
Then
$$\partial_1 F(u,v) = 4u\big(u^2-v^2\big) = 4u^3 - 4uv^2 \qquad \text{and} \qquad \partial_2 F(u,v) = -4v\big(u^2-v^2\big).$$
Moreover, for $(x,y) \in \mathbb{R}^2$,
$$\partial_1 f(x,y) = 2x, \qquad \text{and} \qquad \partial_2 f(x,y) = -8y,$$

so that, for $(u,v)\in\mathbb{R}^2$:

$$\partial_1 f\big(u^2+v^2,uv\big)=2\big(u^2+v^2\big),\qquad\text{and}\qquad \partial_2 f\big(u^2+v^2,uv\big)=-8uv,$$

and hence:

$$\begin{aligned}2u\partial_1 f\big(u^2+v^2,uv\big)+v\partial_2 f\big(u^2+v^2+uv\big)&=4u\big(u^2+v^2\big)-8uv^2\\&=4u^3+4uv^2-8uv^2\\&=4u^3-4uv^2\\&=\partial_1 F(u,v),\end{aligned}$$

and

$$2v\partial_1 f\big(u^2+v^2,uv\big)+u\partial_2 f\big(u^2+v^2,uv\big)=4v\big(u^2+v^2\big)-8u^2v=-4u^2v+4v^3=\partial_2 F(u,v).$$

Exercise 5.

1. a) Let $g\in E$. Then:

$$\big|\varphi(g)\big|=\left|\int_0^1 t\,f_0(t)g(t)\,\mathrm{d}t\right|\le\int_0^1 t\big|f_0(t)\big|\big|g(t)\big|\,\mathrm{d}t\le\int_0^1 t\big|f_0(t)\big|\|g\|_\infty\,\mathrm{d}t\le\|g\|_\infty\int_0^1 t\big|f_0(t)\big|\,\mathrm{d}t\underset{\|g\|_\infty\to0}{\longrightarrow}0,$$

   hence φ is continuous at 0_E and hence φ is continuous.

   b) For $t\in[0,1]$,

$$t\big|h(t)\big|^2\le t\|h\|_\infty^2,$$

   hence

$$\int_0^1 t\big|h(t)\big|^2\,\mathrm{d}t\le\|h\|_\infty^2\int_0^1\,\mathrm{d}t=\frac{1}{2}\|h\|_\infty.$$

2. Let $h\in E$. Then:

$$\begin{aligned}\Phi(f_0+h)&=\int_0^1 t\big(f_0(t)+h(t)\big)^2\,\mathrm{d}t\\&=\int_0^1 t\big(f_0^2(t)+2f_0(t)h(t)+h(t)^2\big)\,\mathrm{d}t\\&=\int_0^1 tf_0^2(t)\,\mathrm{d}t+2\int_0^1 t\,f_0(t)h(t)\,\mathrm{d}t+\int_0^1 t\,h(t)^2\big)\,\mathrm{d}t\\&=\Phi(f_0)+2\varphi(h)+\int_0^1 th(t)^2\,\mathrm{d}t.\end{aligned}$$

   We recognize that the remainder is the term studied in Question 1b, and hence:

$$\frac{1}{\|h\|_\infty}\left|\int_0^1 t\,h(t)^2\,\mathrm{d}t\right|\le\frac{1}{2}\|h\|_\infty\underset{\|h\|_\infty\to0}{\longrightarrow}0.$$

   This (together with the fact that φ is continuous) shows that Φ is differentiable f_0 and that $D_{f_0}\Phi=2\varphi$.

3. The directional derivative of Φ at f_0 in the direction h can be computed by:

$$D_{f_0}\Phi(h)=2\varphi(h)=2\int_0^1 t^4\,\mathrm{d}t=\frac{2}{5}.$$