SCAN 2 — S1 — Math Test #2 — 1h30

November 30, 2020

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

The marks are given as a guide only; the final marking scheme might differ *slightly* from the marks provided here.

Exercise 1 (*3 points*)**.** Let f be the function defined by:

$$f : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & \begin{cases} \dfrac{x^3+y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

Determine whether $\partial_1 f(0,0)$ and $\partial_2 f(0,0)$ exist, and if they exist, determine their value. Is f differentiable at $(0,0)$?

Exercise 2 (*3 points*)**.** Let f be the function defined by:

$$f : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & \begin{cases} \dfrac{x^2y+xy^2}{x^2+2y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

Show that f is continuous.

Exercise 3 (*3 points*)**.** Let $u : \mathbb{R}^2 \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be differentiable functions, and let g be the function defined by:

$$g : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & x\,u\big(xy, 2x + f(y)\big). \end{array}$$

Let $(x,y) \in \mathbb{R}^2$. Compute the first-order partial derivatives of g (you don't need to justify that they exist, only compute them).

Exercise 4 (*5 points*)**.** Let

$$\varphi : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (u,v) & \longmapsto & \big(u^2+v^2, uv\big). \end{array}$$

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function, and define $F = f \circ \varphi$.

1. Let $(u,v) \in \mathbb{R}^2$. Compute the Jacobian matrix $J_{(u,v)}\varphi$ of φ at (u,v).
2. Give a relation between the Jacobian matrices of f, F and φ (each taken at appropriate points you will specify).
3. Use the previous relation to express the first-order partial derivatives of F in terms of that of f (at a point you will specify).
4. Check your answer in the special case where

$$f : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & x^2 - 4y^2. \end{array}$$

Exercise 5 (*6 points*)**.** Let $E = C^1([0,1])$ be the vector space that consists of functions of class C^1 on $[0,1]$. Throughout this exercise, the norm on E is the ∞-norm, denoted by $\|\cdot\|_\infty$.

Let Φ be the function defined by:

$$\begin{array}{rccc} \Phi : & E & \longrightarrow & \mathbb{R} \\ & f & \longmapsto & \displaystyle\int_0^1 t f(t)^2 \,\mathrm{d}t \end{array}.$$

From now on, we fix an element $f_0 \in E$.

We define the following map:

$$\begin{array}{rccc} \varphi : & E & \longrightarrow & \mathbb{R} \\ & g & \longmapsto & \displaystyle\int_0^1 t f_0(t) g(t) \,\mathrm{d}t \end{array}.$$

You're given that φ is linear.

1. *Preliminary questions.*
   a) Show that φ is continuous. You can use the well-known fact that states that a linear map on E is continuous if and only if it is continuous at 0_E.
   b) Let $h \in E$. Show that:
   $$\int_0^1 t\left|h(t)\right|^2 \mathrm{d}t \leq \frac{1}{2}\|h\|_\infty^2.$$
2. Show that Φ is differentiable at f_0 and express its differential $D_{f_0}\Phi$ in terms of φ.
3. In this question, f_0 is the following element of E:
   $$\begin{array}{rccc} f_0 : & [0,1] & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & x^2. \end{array}$$
   Let h be the following element of E:
   $$\begin{array}{rccc} h : & [0,1] & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & x. \end{array}$$
   Give the directional derivative of Φ at f_0 in the direction h.