SCAN 2 — Solution of Math Test #1

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Exercise 1.

- The function $t \mapsto \mathrm{e}^{-t}/\sqrt{t}$ is continuous on $[1,+\infty)$, hence I is improper at $+\infty$. Now,

$$\forall t \in [1,+\infty),\ 0 \leq \frac{\mathrm{e}^{-t}}{\sqrt{t}} \leq \mathrm{e}^{-t},$$

  and we know that the improper integral

$$\int_1^{+\infty} \mathrm{e}^{-t}\,\mathrm{d}t$$

  is convergent; hence, by the comparison test, I converges.

- The function $t \mapsto t^\alpha/\big((1+t)\sqrt{t}\big)$ is continuous on $(0,+\infty)$, hence J_α is improper at 0^+ and at $+\infty$.

  - Convergence at 0^+: since

$$\frac{t^\alpha}{(1+t)\sqrt{t}} \underset{t\to 0^+}{\sim} \frac{1}{t^{1/2-\alpha}} > 0.$$

    we conclude, by the equivalent test (and by Riemann at a finite point) that J_α converges at 0^+ if and only if $1/2-\alpha < 1$ i.e., $\alpha > -1/2$.

  - Convergence at $+\infty$: since

$$\frac{t^\alpha}{(1+t)\sqrt{t}} \underset{t\to 0^+}{\sim} \frac{1}{t^{3/2-\alpha}} > 0.$$

    we conclude, by the equivalent test (and by Riemann at $+\infty$) that J_α converges at $+\infty$ if and only if $3/2-\alpha > 1$ i.e., $\alpha < 1/2$.

  Conclusion: J_α converges if and only if $\alpha \in (-1/2, 1/2)$.

Exercise 2.

1. Let $x \in \mathbb{R}_+^*$. Since $x^3 > 0$, the function $t \mapsto 1/\big(x^3+t^3\big)$ is continuous on $[0,+\infty)$, hence the improper integral

$$I_x = \int_0^{+\infty} \frac{\mathrm{d}t}{x^3+t^3}$$

   is improper at $+\infty$. Now,

$$\frac{1}{x^3+t^3} \underset{t\to+\infty}{\sim} \frac{1}{t^3} > 0$$

   hence, by the equivalent test (and by Riemann at $+\infty$ with $\alpha > 3$), we conclude that I_x converges.

   Hence f is well defined.

2. Let $x \in \mathbb{R}_+^*$. Since

$$\forall t \in [0,+\infty),\ \frac{1}{x^3+t^3} \geq 0$$

   (and the endpoints of the integral are in the correct order) we conclude that $f(x) \geq 0$. In fact $f(x) > 0$ since the function we're integrating is non-negative, non-nil and continuous.

3. Let $x_1, x_2 \in \mathbb{R}_+^*$ such that $x_1 < x_2$. Then:

$$\forall t \in [0,+\infty),\ \frac{1}{x_1^3+t^3} > \frac{1}{x_2^3+t^3}$$

   and integrating from 0 to $+\infty$ yields $f(x_1) > f(x_2)$ (the strict inequality is obtained by observing that the functions we're integrating are ordered, distinct and continuous). Hence f is decreasing.

4. Let $A > 0$. By using the substitution $u = t/x$ we obtain:

$$\int_0^A \frac{\mathrm{d}t}{t^3 + x^3} = \int_0^{A/x} \frac{x\,\mathrm{d}u}{x^3 + x^3u^3} = \frac{1}{x^2}\int_0^{A/x} \frac{\mathrm{d}u}{1 + u^3}$$

hence (taking the limit as $A \to +\infty$) yields:

$$f(x) = \frac{1}{x^2} f(1).$$

5. See Figure 1.

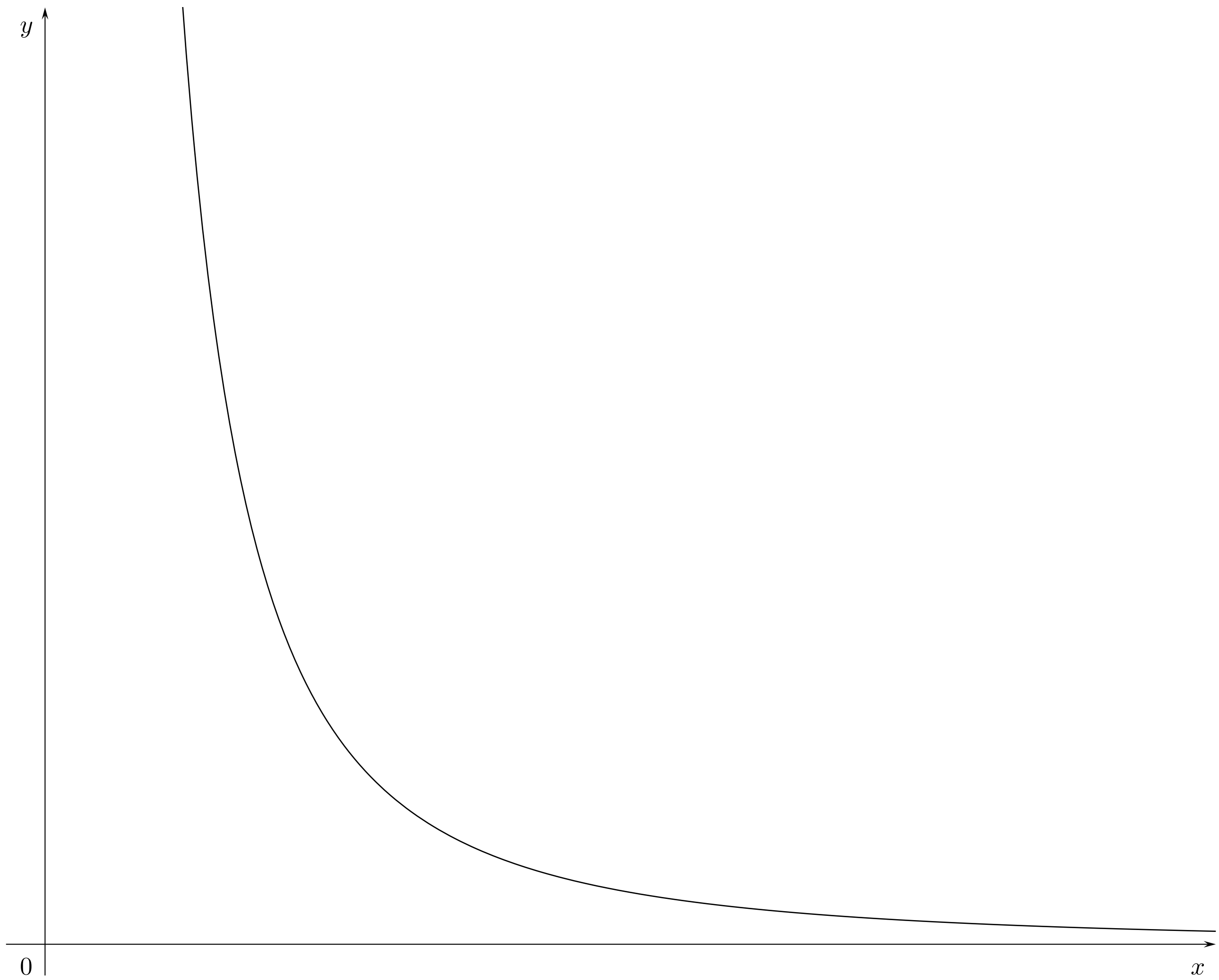

Figure 1. Graph of the function f of Exercise 2: f is decreasing and positive. In fact, this graph is of the form $y = f(1)/x^2$ with $f(1) > 0$.

Exercise 3.

1. We check that N satisfies the conditions to be a norm: [1]

   - Separation property: let $P \in E$ such that $N(P) = 0$. Then, since the function $t \mapsto tP(t)^2$ is non-negative and continuous on $[0,1]$, we conclude that
   $$\forall t \in [0,1],\ tP(t)^2 = 0,$$
   hence
   $$\forall t \in (0,1],\ P(t) = 0.$$
   Then the polynomial P has an infinite number of roots, and hence $P = 0_E$.
   - Positive homogeneity: let $P \in E$ and $\lambda \in \mathbb{R}$. Then:
   $$N(\lambda P) = \left(\int_0^1 t\lambda^2 P(t)^2 \,\mathrm{d}t\right)^{1/2} = \left(\lambda^2 \int_0^1 tP(t)^2 \,\mathrm{d}t\right)^{1/2} = |\lambda| \left(\int_0^1 tP(t)^2 \,\mathrm{d}t\right)^{1/2} = |\lambda| N(P).$$
   - Triangle inequality: let $P, Q \in E$. Define:
   $$f : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \sqrt{t}\,P(t) \end{array} \qquad \text{and} \qquad g : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \sqrt{t}\,Q(t). \end{array}$$
   Then $f, g \in C\big([0,1]\big)$, and $N(P+Q) = \|f+g\|_2$, and since $\|\cdot\|_2$ is a norm on $C\big([0,1]\big)$, we have:
   $$N(P+Q) = \|f+g\|_2 \le \|f\|_2 + \|g\|_2 = N(P) + N(Q).$$

2. $$N(1-X)^2 = \int_0^1 t(1-t)^2 \,\mathrm{d}t = \int_0^1 \big(t^3 - 2t^2 + t\big) \,\mathrm{d}t = \frac{1}{4} - \frac{2}{3} + \frac{1}{2} = \frac{1}{12},$$
   hence the distance between 1 and X with respect to N is:
   $$N(1-X) = \frac{1}{\sqrt{12}} = \frac{1}{2\sqrt{3}}.$$
   The distance between 1 and X with respect to the norm $\|\cdot\|$ is:
   $$\|1-X\| = \int_0^1 |1-t| \,\mathrm{d}t = \int_0^1 (1-t) \,\mathrm{d}t = 1 - \frac{1}{2} = \frac{1}{2}.$$

3. Let $n \in \mathbb{N}$. Then:
   $$\|P_n - 0_E\| = \|P_n\| = \int_0^1 \sqrt{n} t^n \,\mathrm{d}t = \frac{\sqrt{n}}{n+1} \underset{n\to+\infty}{\longrightarrow} 0,$$
   hence the sequence $(P_n)_{n\ge 0}$ converges to 0_E with respect to the norm $\|\cdot\|$.
   $$N(P_n - 0_E)^2 = N(P_x)^2 = \int_0^1 tnt^{2n} \,\mathrm{d}t = n \int_0^1 t^{2n+1} \,\mathrm{d}t = \frac{n}{2n+2} \underset{n\to+\infty}{\longrightarrow} \frac{1}{2} \ne 0,$$
   hence the sequence $(P_n)_{n\ge 0}$ doesn't converge to 0_E with respect to the norm N.

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[1] Another possibility to show that N is a norm: define
$$\varphi : \begin{array}{ccc} E & \longrightarrow & C\big([0,1]\big) \\ P & \longmapsto & \left(\begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \sqrt{t}\,P(t) \end{array}\right). \end{array}$$
Then φ is linear and for $P \in E$, $N(P) = \big\|\varphi(P)\big\|_2$. In order to conclude that N is a norm, we only need to check that φ is injective: let $P \in E$ such that $\varphi(P) = 0$, i.e.,
$$\forall t \in [0,1],\ \sqrt{t}\,P(t) = 0.$$
Then:
$$\forall t \in (0,1],\ P(t) = 0,$$
and hence the polynomial P has an infinite number of roots, hence $P = 0_E$.

4. a) Proposition P1 is false: if it were true, there would exist $\alpha > 0$ such that $\alpha N \geq \|\cdot\|$, and in particular we would have, for $n \in \mathbb{N}$,

$$\alpha N(P_n) = \alpha \frac{\sqrt{n}}{\sqrt{2n+2}} \leq \|P_n\| = \frac{\sqrt{n}}{n+1},$$

and taking the limit as $n \to +\infty$ would yield:

$$\alpha \frac{1}{\sqrt{2}} \leq 0$$

which is impossible since $\alpha > 0$.

b) The norms N and $\|\cdot\|$ are not equivalent.

Exercise 4.

1. Let $n \geq 2$. The function $t \mapsto \ln(t)/(1+t)^n$ is continuous on $[1, +\infty)$, hence I_n is improper at $+\infty$. Now since $n \geq 2$ (and since ln is weaker than polynomials),

$$\lim_{t \to +\infty} \frac{\ln(t) t^{3/2}}{(1+t)^n} = 0$$

hence there exists $A > 1$ such that:

$$\forall t \geq A,\ 0 \leq \frac{\ln(t)}{(1+t)^n} \leq \frac{1}{t^3/2}.$$

Now, the improper integral $\int_1^{+\infty} \frac{\mathrm{d}t}{t^{3/2}}$ is convergent (Riemann at $+\infty$ with $\alpha = 3/2 > 1$), and we conclude, by the Comparison Test, that the improper integral I_n converges.

2. Let $A > 1$. Then, by an integration by parts with

$$f(t) = \ln(t), \qquad f'(t) = \frac{1}{t}$$
$$g(t) = -\frac{1}{(n-1)(1+t)^{n-1}}, \qquad g'(t) = \frac{1}{(1+t)^n}.$$

we obtain:

$$\begin{aligned}\int_1^A \frac{\ln(t)}{(1+t)^n}\,\mathrm{d}t &= \left[-\frac{\ln(t)}{(n-1)(1+t)^{n-1}}\right]_{t=1}^{t=A} + \int_1^A \frac{\mathrm{d}t}{(n-1)t(1+t)^{n-1}} \\ &= -\frac{\ln(A)}{(n-1)(1+A)^{n-1}} + \frac{1}{n-1}\int_1^A \frac{\mathrm{d}t}{t(1+t)^{n-1}}\end{aligned}$$

Finally, taking the limit as $A \to +\infty$ yields:

$$I_n = 0 + \frac{1}{n-1}\int_1^{+\infty} \frac{\mathrm{d}t}{t(1+t)^{n-1}},$$

so that $a_n = 1/(n-1)$.

3. Let $n \geq 3$. Then:

$$\forall t \geq 1,\ 0 \leq \frac{1}{t(1+t)^{n-1}} \leq \frac{1}{(1+t)^{n-1}}$$

hence (the endpoints of the integral being in the correct order):

$$0 \leq \int_1^{+\infty} \frac{\mathrm{d}t}{t(1+t)^{n-1}} \leq \int_1^{\infty} \frac{\mathrm{d}t}{(1+t)^{n-1}} = \frac{1}{(n-2)2^{n-2}}$$

and hence

$$0 \leq I_n = \frac{1}{n-1}\int_1^{+\infty} \frac{\mathrm{d}t}{t(1+t)^{n-1}} \leq \frac{1}{(n-1)(n-2)2^{n-2}}$$

4. a) Let $n \geq 3$ and $t \in (0, +\infty)$. Since $1 + t \neq 1$ we can use the formula for the sum of a geometric progression of ratio $q = 1/(1+t)$:

$$\sum_{k=1}^{n-2} \frac{1}{(1+t)^{k+1}} = \frac{q^2 - q^n}{1-q} = \frac{1/(1+t)^2 - 1/(1+t)^n}{t/(1+t)} = \frac{1}{t(1+t)} - \frac{1}{t(1+t)^{n-1}}$$

b) We differentiate F_n: for $t \in (0, +\infty)$,

$$\begin{aligned}
F_n'(t) &= \frac{1}{t} - \frac{1}{1+t} + \sum_{k=1}^{n-2} -\frac{k}{k(1+t)^{k+1}} \\
&= \frac{1}{t} - \frac{1}{1+t} - \sum_{k=1}^{n-2} \frac{1}{(1+t)^{k+1}} \\
&= \frac{1}{t} - \frac{1}{1+t} - \frac{1}{t(1+t)} + \frac{1}{t(1+t)^{n-1}} && \textit{By the previous question} \\
&= \frac{1+t-t}{t(1+t)} - \frac{1}{t(1+t)} + \frac{1}{t(1+t)^{n-1}} \\
&= \frac{1}{t(1+t)^{n-1}} = f_n(t).
\end{aligned}$$

5. Let $n \geq 3$. Then:

$$\begin{aligned}
(n-1)I_n &= \int_1^{+\infty} \frac{\mathrm{d}t}{t(1+t)^{n-1}} && \textit{by Question 2} \\
&= \lim_{A \to +\infty} F_n(A) - F_n(1) && \textit{by Question 4b)} \\
&= \lim_{A \to +\infty} \ln\left(\frac{A}{1+A}\right) + \sum_{k=1}^{n-2} \frac{1}{k(1+A)^k} - \ln\left(\frac{1}{2}\right) - \sum_{k=1}^{n-2} \frac{1}{k2^k} \\
&= 0 + 0 - \ln\left(\frac{1}{2}\right) - \sum_{k=1}^{n-2} \frac{1}{k2^k} \\
&= \ln 2 - \sum_{k=1}^{n-2} \frac{1}{k2^k}
\end{aligned}$$

Hence $C = \ln 2$.

6. By Question 3,

$$\forall n \geq 3,\ 0 \leq I_n \leq \frac{1}{(n-2)2^{n-2}}$$

and we conclude, by the Squeeze Theorem, that $\lim\limits_{n \to +\infty} (n-1)I_n = 0$.

Hence, taking the limit as $n \to +\infty$ in the relation obtained in Question 5 yields:

$$0 = \ln 2 - \lim_{n \to +\infty} \sum_{k=1}^{n-2} \frac{1}{k2^k}$$

from which we conclude that $\ell = \ln 2$.

Exercise 5. P3 is true: if $u \in \overline{B}_N$ then $N(u) \leq 1$, i.e., $\|u\| + \|u\|' \leq 1$. Since $\|u\| \geq 0$ and $\|u\|' \geq 0$ we conclude that $\|u\| \leq 1$ and $\|u\|' \leq 1$, hence $u \in \overline{B}$ and $u \in \overline{B}'$, and we conclude that $u \in \overline{B} \cap \overline{B}'$.

If we replace N with N' the result of $P3$ is still true, namely $\overline{B}'_N \subset \overline{B} \cap \overline{B}'$: the same proof as above applies in this case, as the crucial step

$$\max\{\|u\|, \|u'\|\} \leq 1 \implies \|u\| \leq 1 \text{ and } \|u\|' \leq 1$$

is still valid.

Exercise 6.

- Case $\alpha > 3$. Since $\mathbb{R}^3$ is a finite dimensional vector space, all norms on $\mathbb{R}^3$ are equivalent, and we choose to use the 4-norm. Let $(x, y, z) \in \mathbb{R}^3 \setminus \mathbf{0}$. Then:

$$\begin{aligned}\left|f(x,y,z) - f(\mathbf{0})\right| &= \frac{|x|^\alpha |z|}{x^4 + y^4 + z^4}\\ &\le \frac{\left\|(x,y,z)\right\|_4^{\alpha+1}}{\left\|(x,y,z)\right\|_4^4}\\ &\le \left\|(x,y,z)\right\|_4^{\alpha-3}\\ &\underset{(x,y,z)\to\mathbf{0}}{\longrightarrow} 0.\end{aligned}$$

  hence $\lim\limits_{(x,y,z)\to\mathbf{0}} f(x,y,z) = 0$.

- Case $\alpha \le 3$. Let $t \in \mathbb{R}^*$. Then:

$$\left|f(t,t,t)\right| = \frac{|t|^\alpha |t|}{3t^4} = \frac{1}{3}|t|^{\alpha-3} \underset{t\to 0}{\longrightarrow} \begin{cases} 1/3 & \text{if } \alpha = 3\\ +\infty & \text{if } \alpha < 3\end{cases}$$

  and this limit is not nil, yet

$$\lim_{t\to 0}(t,t,t) = (0,0,0) = \mathbf{0}.$$

  We hence conclude, by the Composition of Limits Theorem, that f is not continuous at $\mathbf{0}$.