INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Math Test #1 — 2h *October 18, 2021*

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

The marks are given as a guide only; the final marking scheme might differ *slightly* from the marks provided here.

Exercises 1, 2 and 4 are common with PCC2.

Exercise 1 (*1.25 points*)**.** Study the convergence of the following improper integrals:

$$I = \int_1^{+\infty} \frac{\mathrm{e}^{-t}}{\sqrt{t}}\,\mathrm{d}t \quad \text{and, for } \alpha \in \mathbb{R},\ J_\alpha = \int_0^{+\infty} \frac{t^\alpha}{(1+t)\sqrt{t}}\,\mathrm{d}t.$$

For J_α, your answer will depend on the value of $\alpha \in \mathbb{R}$.

Exercise 2 (*3.5 points*)**.** Let

$$f : \begin{array}{ccc} \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \displaystyle\int_0^{+\infty} \frac{\mathrm{d}t}{x^3+t^3}. \end{array}$$

1. Show that the function f is well defined.
2. For $x \in \mathbb{R}_+^*$, determine the sign of $f(x)$.
3. Determine the variations of the function f. *Do not compute the derivative of f.*
4. For $x \in \mathbb{R}_+^*$, use the substitution $u = t/x$ to find a relation between $f(x)$ and $f(1)$.
5. Sketch the graph of f. The information obtained in the previous questions should appear in your figure. *Do not try to compute the value of $f(1)$.*

Exercise 3 (*5 points*)**.** Let $E = \mathbb{R}[X]$ and define:

$$N : \begin{array}{ccc} E & \longrightarrow & \mathbb{R}_+ \\ P & \longmapsto & \left(\displaystyle\int_0^1 tP(t)^2\,\mathrm{d}t\right)^{1/2} \end{array} \qquad \text{and} \qquad \|\cdot\| : \begin{array}{ccc} E & \longrightarrow & \mathbb{R}_+ \\ P & \longmapsto & \displaystyle\int_0^1 \left|P(t)\right|\mathrm{d}t. \end{array}$$

You're given that N is well defined and you don't need to justify this fact.

1. Show that N is a norm on E. We admit that $\|\cdot\|$ is also a norm on E.
2. Compute the distance between 1 and X with respect to N and with respect to $\|\cdot\|$.
3. Define:

   $$\forall n \geq 0,\ P_n = \sqrt{n}\,X^n.$$

   Show that the sequence $(P_n)_{n\geq 0}$ converges to 0_E for the norm $\|\cdot\|$, but not for the norm N.
4. a) From the previous questions, you can deduce that one of the following two propositions is *false.* Which one is it? justify your answer.

   P1. There exists $\alpha > 0$ such that $\alpha N \leq \|\cdot\|$ **P2.** There exists $\beta > 0$ such that $\|\cdot\| \leq \beta N$.

   b) What can you conclude about the norms N and $\|\cdot\|$?

Exercise 4 (*7.25 points*)**.** Define:

$$\forall n \in \mathbb{N},\ n \geq 2, \qquad I_n = \int_1^{+\infty} \frac{\ln(t)}{(1+t)^n}\,\mathrm{d}t.$$

1. Show that for all $n \geq 2$ the improper integral I_n converges.
2. Let $n \geq 2$. Use an integration by parts to determine the number $a_n \in \mathbb{R}$ such that:
$$I_n = a_n \int_1^{+\infty} \frac{\mathrm{d}t}{t(1+t)^{n-1}}.$$
3. Show that:
$$\forall n \geq 3, \qquad 0 \leq I_n \leq \frac{1}{(n-1)(n-2)2^{n-2}}.$$
4. a) Show that:
$$\forall n \geq 3,\ \forall t \in (0,+\infty), \quad \sum_{k=1}^{n-2} \frac{1}{(1+t)^{k+1}} = \frac{1}{t(1+t)} - \frac{1}{t(1+t)^{n-1}}.$$
Hint: sum of a geometric progression.

   b) Let $n \geq 3$ and define:
$$f_n : \begin{array}{ccc} (0,+\infty) & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \dfrac{1}{t(1+t)^{n-1}} \end{array} \quad \text{and} \quad F_n : \begin{array}{ccc} (0,+\infty) & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \ln(t) - \ln(1+t) + \displaystyle\sum_{k=1}^{n-2} \frac{1}{k(1+t)^k}. \end{array}$$
Show that the function F_n is an antiderivative of the function f_n.
5. Deduce that there exists a constant $C \in \mathbb{R}$ that you will determine such that:
$$\forall n \geq 3, \quad (n-1)I_n = C - \sum_{k=1}^{n-2} \frac{1}{k2^k}.$$
6. Deduce that the following limit exists in $\mathbb{R}$, and determine its value:
$$\ell = \lim_{N\to+\infty} \sum_{k=1}^{N} \frac{1}{k2^k}.$$

Exercise 5 (*1.5 marks*)**.** Let E be a vector space and let $\|\cdot\|$ and $\|\cdot\|'$ be two norms on E, and denote by $\overline{B}$ and $\overline{B}'$ their respective closed unit balls. You're given that $N = \|\cdot\| + \|\cdot\|'$ is a norm on E, the closed unit ball of which is denoted by $\overline{B}_N$.
Only one of the following propositions is true. Which one is it? justify your answer.

P1. $\overline{B} \cup \overline{B}' \subset \overline{B}_N$ **P2.** $\overline{B} \cap \overline{B}' \subset \overline{B}_N$ **P3.** $\overline{B}_N \subset \overline{B} \cap \overline{B}'$.

Is the same true if we replace N with $N' = \max\{\|\cdot\|, \|\cdot\|'\}$?

Exercise 6 (*1.5 points*)**.** Let $\alpha \in \mathbb{R}$ and define:

$$f : \begin{array}{ccc} \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ (x,y,z) & \longmapsto & \begin{cases} \dfrac{|x|^\alpha z}{x^4+y^4+z^4} & \text{if } (x,y,z) \neq \mathbf{0} \\ 0 & \text{otherwise} \end{cases} \end{array}.$$

Show that f is continuous at $\mathbf{0} = (0,0,0)$ if and only if $\alpha > 3$.
Hint 1. You may choose an appropriate p-norm on $\mathbb{R}^3$ (explain why you can).
Hint 2. For the case $\alpha \leq 3$, you will choose an appropriate path along which you will compute the limit.