INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Solution of Math Test #2

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Exercise 1.

1. Let $u \in E$. For $h \in E$ one has:

$$\begin{aligned} q(u+h) &= \int_0^1 u(t)^2 \,\mathrm{d}t + 2\int_0^1 u(t)\,h(t)\,\mathrm{d}t + \int_0^1 h(t)^2\,\mathrm{d}t \\ &= q(u) \ + \ 2\int_0^1 u(t)\,h(t)\,\mathrm{d}t \ + \ \int_0^1 h(t)^2\,\mathrm{d}t. \end{aligned}$$

In this expression, we identify the constant term as $q(u)$, the linear term (with respect to h) as the second term, and the remainder as the last term.

- We now check for the continuity of the linear map

$$\begin{aligned} \alpha \ : \ E &\longrightarrow \mathbb{R} \\ h &\longmapsto \int_0^1 u(t)\,h(t)\,\mathrm{d}t. \end{aligned}$$

Let $h \in E$:

$$\begin{aligned} \bigl|\alpha(h)\bigr| &= \left|\int_0^1 u(t)\,h(t)\,\mathrm{d}t\right| \\ &\le \int_0^1 \bigl|u(t)\bigr|\bigl|h(t)\bigr|\,\mathrm{d}t && \textit{Triangle inequality for integrals, } 0 \le 1 \\ &\le \int_0^1 \bigl|u(t)\bigr|\|h\|_\infty\,\mathrm{d}t && \textit{since } |h| \le \|h\|_\infty \\ &= \|h\|_\infty \int_0^1 \bigl|u(t)\bigr|\,\mathrm{d}t. \end{aligned}$$

Hence $\alpha(h) \underset{\|h\|_\infty \to 0}{\longrightarrow} 0 = \alpha(0_E)$, hence α is continuous at 0_E, hence (since α is linear) α is continuous.

- We now check that the remainder is a $o\bigl(\|h\|_\infty\bigr)$ as $\|h\|_\infty \to 0$:

$$\left|\int_0^1 h(t)^2\,\mathrm{d}t\right| \le \int_0^1 \|h\|_\infty^2\,\mathrm{d}t = \|h\|_\infty^2 \underset{\|h\|_\infty \to 0}{=} o\bigl(\|h\|_\infty\bigr).$$

Hence q is differentiable at u and $\mathrm{d}_u q = \alpha(u)$.

2. Let $u \in E$ and $t \in [0,1]$. Then:

$$\begin{aligned} \bigl|\varphi(u)(t)\bigr| &\le \left|\omega \mathrm{e}^{-\omega t}\int_0^t u(s)\mathrm{e}^{\omega s}\,\mathrm{d}s\right| \\ &\le \|u\|_\infty \left|\omega \mathrm{e}^{-\omega t}\int_0^t \mathrm{e}^{\omega s}\,\mathrm{d}s\right| \\ &= \|u\|_\infty \left|\mathrm{e}^{-\omega t}\bigl[\mathrm{e}^{\omega s}\bigr]_{s=0}^{s=t}\right| \\ &= \|u\|_\infty \bigl|\mathrm{e}^{-\omega t}\bigl(\mathrm{e}^{\omega t} - 1\bigr)\bigr| \\ &= \|u\|_\infty \bigl(1 - \mathrm{e}^{-\omega t}\bigr) \\ &\le \|u\|_\infty \bigl(1 - \mathrm{e}^{-\omega}\bigr). \end{aligned}$$

Hence

$$\bigl\|\varphi(u)\bigr\| \le \|u\|_\infty \bigl(1 - \mathrm{e}^{-\omega}\bigr).$$

From the previous inequality we conclude that φ is continuous at 0_E, and since φ is linear we conclude that φ is continuous.

3. Let $u \in E$. Then:

$$\begin{aligned}
\|\psi(u)\| &= \omega\|u-\varphi(u)\|_\infty \\
&\le \omega\Big(\|u\|_\infty + \|\varphi(u)\|_\infty\Big) && \textit{by the triangle inequality} \\
&\le \omega\Big(\|u\|_\infty + \|u\|_\infty\big(1-\mathrm{e}^{-\omega}\big)\Big) && \textit{from Question 2} \\
&= \omega\big(2-\mathrm{e}^{-\omega}\big)\|u\|_\infty.
\end{aligned}$$

Hence $K = \omega\big(2-\mathrm{e}^{-\omega}\big)$ is such a value of K.

From the previous inequality, we conclude that ψ is continuous at 0_E. Since ψ is linear, we conclude that ψ is continuous. Hence ψ is a linear continuous map, and we can conclude that ψ is differentiable and that:

$$\forall u \in E,\ \mathrm{d}_u\psi = \psi.$$

4. Notice that:

$$\forall u \in E,\ W(u) = \int_0^1 \left(\frac{1}{\omega}\psi(u)(t)\right)^2 \mathrm{d}t = \frac{1}{\omega^2}q\big(\psi(u)\big),$$

hence

$$W = \frac{1}{\omega^2}q \circ \psi.$$

Since ψ and q are differentiable we conclude, by the Chain Rule, that W is differentiable and that:

$$\begin{aligned}
\forall u \in E,\ \mathrm{d}_uW &= \frac{1}{\omega^2}\mathrm{d}_{\psi(u)}q \circ D_u\psi \\
&= \frac{1}{\omega^2}\mathrm{d}_{\psi(u)}q \circ \psi.
\end{aligned}$$

More explicitly, for $u \in E$,

$$\begin{aligned}
\mathrm{d}_uW\ :\ E &\longrightarrow \mathbb{R} \\
h &\longmapsto \frac{1}{\omega^2}\int_0^1 \psi(u)(t)\,\psi(h)(t)
\end{aligned}.$$

Exercise 2.

1. Let $v \in E$, and define:

$$\begin{aligned}
\varphi\ :\ \mathbb{R} &\longrightarrow \mathbb{R} \\
t &\longmapsto f(P+tv).
\end{aligned}$$

Then, for $t \in \mathbb{R}$:

$$\begin{aligned}
\varphi(t) &= P(0) + tv(0) + \big(P+tv\big)'(1)^2 \\
&= P(0) + tv(0) + \big(P'(1)+tv'(1)\big)^2 \\
&= P(0) + P'(1)^2 + t\big(v(0)+2P'(1)v'(1)\big) + t^2h'(1)^2,
\end{aligned}$$

so that $\varphi'(0) = v(0) + 2P'(1)v'(1)$. This shows that $\nabla_v f(P)$ exists. We finally conclude:

$$\forall v \in E,\ \nabla_v f(P) = v(0) + 2P'(1)v'(1).$$

2. In this case: $v(0) = 2$, $v' = -1 + 2X$ so that $v'(1) = 1$ and $P_0' = -2X$ so that $P_0'(1) = -2$. Hence:

$$\nabla_v f(P_0) = -2.$$

3. Assuming that f is differentiable at P, we have:

$$\forall v \in E,\ \mathrm{d}_P f(v) = \nabla_v P,$$

hence:

$$\begin{aligned}
\mathrm{d}_P f\ :\ E &\longrightarrow \mathbb{R} \\
v &\longmapsto v(0) + 2P'(1)v'(1).
\end{aligned}$$

Exercise 3. Notice that f can be written as:

$$f : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & \begin{cases} \dfrac{x^4}{x^2+y^2} & \text{if } (x,y)\in U \\ 0 & \text{otherwise} \end{cases} \end{array}$$

with $U = \mathbb{R}^2 \setminus \{(0,0)\}$.

1. By elementary operations, f is continuous on $\mathring{U} = U$. We now check that continuity of f at $(0,0)$: let $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$:

$$\left|f(x,y) - f(0,0)\right| = \left|\frac{x^4}{x^2+y^2}\right| \leq \frac{\|(x,y)\|_2^4}{\|(x,y)\|_2^2} = \|(x,y)\|_2^2 \underset{\|(x,y)\|_2\to 0}{\longrightarrow} 0.$$

   Hence f is also continuous at $(0,0)$.

2. Let $v = (a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Then, for $t \in \mathbb{R}^*$:

$$\frac{f(tv) - f(0,0)}{t} = \frac{1}{t}\frac{t^4a^4}{t^2(a^2+b^2)} = t\frac{a^4}{a^2+b^2} \underset{t\to 0}{\longrightarrow} 0.$$

   Hence $\nabla_v f(0,0) = 0$.

3. We know that the first-order partial derivatives of f at $(0,0)$ are directional derivative of f at $(0,0)$, hence the first-partial derivatives of f at $(0,0)$ exist and we have:

$$\partial_1 f(0,0) = \nabla_{e_1} f(0,0) = 0 \qquad \text{and} \qquad \partial_2 f(0,0) = \nabla_{e_2} f(0,0) = 0.$$

4. Let $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Then:

$$\partial_1 f(x,y) = \frac{2x^3\big(x^2+2y^2\big)}{\big(x^2+y^2\big)^2} \qquad \text{and} \qquad \partial_2 f(x,y) = -\frac{2x^4y}{\big(x^2+y^2\big)^2}.$$

   Then:

$$\left|\partial_1 f(x,y)\right| = \left|\frac{2x^3\big(x^2+2y^2\big)}{\big(x^2+y^2\big)^2}\right| \leq 2\frac{\|(x,y)\|_2^3 \times 2\|(x,y)\|_2^2}{\|(x,y)\|_2^4} = 4\|(x,y)\|_2 \underset{\|(x,y)\|_2\to 0}{\longrightarrow} 0$$

   and

$$\left|\partial_1 f(x,y)\right| = \left|-\frac{2x^4y}{\big(x^2+y^2\big)^2}\right| \leq 2\frac{\|(x,y)\|_2^5}{\|(x,y)\|_2^4} = 2\|(x,y)\|_2 \underset{\|(x,y)\|_2\to 0}{\longrightarrow} 0$$

   Hence $\partial_1 f$ and $\partial_2 f$ are continuous at $(0,0)$.

5. Since $\partial_1 f$ and $\partial_2 f$ exist in a neighborhood of $(0,0)$ and are continuous at $(0,0)$, we conclude that f is differentiable at $(0,0)$. Moreover:

$$\mathrm{d}_{(0,0)} f = \partial_1 f(0,0)e_1' + \partial_2 f(0,0)e_2' = \mathbf{0}_{\mathscr{L}(\mathbb{R}^2,\mathbb{R})}$$

   (it's the nil linear form on $\mathbb{R}^2$).

Exercise 4.

1. Let $(x,y) \in \mathbb{R}^2$. Then:

$$\begin{aligned} \partial_1 g(x,y) &= y\,\partial_1 f\big(xy,\,-x,\,y^2\big) \;-\; \partial_2 f\big(xy,\,-x,\,y^2\big) \\ \partial_2 g(x,y) &= x\,\partial_1 f\big(xy,\,-x,\,y^2\big) \;+\; 2y\,\partial_3 f\big(xy,\,-x,\,y^2\big). \end{aligned}$$

2. Since f is obtained from elementary operations and the function sin (which is of class C^1), we conclude that f is of class C^1. More explicitly: for $(x,y) \in \mathbb{R}^2$,

$$\partial_1 f(x,y) = \sin(y) \qquad \text{and} \qquad \partial_2 f(x,y) = x\cos(y) + 2y^2$$

from which we notice that $\partial_1 f$ and $\partial_2 f$ are continuous. From these partial derivatives, we can write:

$$\mathrm{d}_{(x,y)} f = \sin(y) e_1' \ + \big(x\cos(y) + 2y^2\big) e_2'$$

or, equivalently,

$$\begin{array}{rccl} \mathrm{d}_{(x,y)} f \ : & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (h_1, h_2) & \longmapsto & \sin(y) h_1 \ + \ \big(x\cos(y) + 2y^2\big) h_2. \end{array}$$

3. Define the path $p : t \mapsto (t,t)$. Then $\lim\limits_{t\to 0} p(t) = (0,0)$ and, for $t \in \mathbb{R}^*$:

$$f\big(p(t)\big) = f(t,t) = \frac{t^2}{2t^2} = \frac{1}{2} \underset{t\to 0}{\longrightarrow} \frac{1}{2} \neq f\big(p(0)\big) = 0,$$

hence f is not continuous at $(0,0)$. We now compute the first-order partial derivatives of f at $(0,0)$:

$$\frac{f(t,0) - f(0,0)}{t} = \frac{0-0}{t} = 0 \underset{t\to 0}{\longrightarrow} 0$$

and

$$\frac{f(0,t) - f(0,0)}{t} = \frac{0-0}{t} = 0 \underset{t\to 0}{\longrightarrow} 0,$$

hence $\partial_1 f(0,0)$ and $\partial_2 f(0,0)$ exist and:

$$\partial_1 f(0,0) = \partial_2 f(0,0) = 0.$$