INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Math Test #2 — 2h

November 29, 2021

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. Let $E = C^1([0,1])$, $\omega \in \mathbb{R}_+^*$. Throughout this exercise, E is equipped with the ∞-norm, denoted by $\|\cdot\|_\infty$. We define:

$$\begin{aligned} \varphi : E &\longrightarrow E \\ u &\longmapsto \left(t \mapsto \omega e^{-\omega t}\int_0^t u(s)e^{\omega s}\,ds\right). \end{aligned}$$

You're given that φ is well-defined, linear, and that for $u \in E$, $\varphi(u)$ is the unique solution to the following initial value problem:

$$\begin{cases} \varphi(u)(0) = 0 \\ \forall t \in [0,1],\ \dfrac{1}{\omega}\varphi(u)'(t) + \varphi(u)(t) = u(t). \end{cases}$$

The differential equation can also be written in point-free form:

$$\frac{1}{\omega}\varphi(u)' + \varphi(u) = u.$$

We also define:

$$\begin{aligned} W : E &\longrightarrow \mathbb{R} \\ u &\longmapsto \int_0^1 \big(u(t) - \varphi(u)(t)\big)^2\,dt. \end{aligned}$$

1. Preliminary question: show that the function

$$\begin{aligned} q : E &\longrightarrow \mathbb{R} \\ u &\longmapsto \int_0^1 u(t)^2\,dt \end{aligned}$$

   is differentiable, and that for all $u \in E$,

$$\begin{aligned} d_u q : E &\longrightarrow \mathbb{R} \\ h &\longmapsto 2\int_0^1 u(t)\,h(t)\,dt \end{aligned}.$$

2. Check that:

$$\forall u \in E,\ \|\varphi(u)\|_\infty \le \|u\|_\infty (1 - e^{-\omega}).$$

   Deduce that φ is continuous.

3. We define

$$\begin{aligned} \psi : E &\longrightarrow E \\ u &\longmapsto \varphi(u)' = \omega(u - \varphi(u)), \end{aligned}$$

   i.e., $\psi = \omega(\mathrm{id}_E - \varphi)$.
   Show that there exists $K \in \mathbb{R}_+$ (and determine such a K) such that

$$\forall u \in E,\ \|\psi(u)\|_\infty \le K\|u\|_\infty.$$

   Deduce that ψ is differentiable and, for $u \in E$, determine $D_u\psi$.

4. Show, using the Chain Rule, that W is differentiable and, for $u \in E$, determine $d_u W$.

Exercise 2. Let $E = \mathbb{R}[X]$ and let

$$\begin{array}{rccl} f: & E & \longrightarrow & \mathbb{R} \\ & P & \longmapsto & P(0) + P'(1)^2. \end{array}$$

Let $P \in E$.

1. Show that all the directional derivatives of f at P exist, and determine them.
2. Let $v = 2 - X + X^2$ and $P_0 = 1 - X^2$. Determine the value of the directional derivative $\nabla_v f(P_0)$ of f at P_0.
3. Assume that f is differentiable (with respect to a certain norm $\|\cdot\|$) at P. Determine $d_P f$.

Exercise 3. Let

$$\begin{array}{rccl} f: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & \begin{cases} \dfrac{x^4}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases} \end{array}$$

1. Show that f is continuous.
2. Show that for all $v \in \mathbb{R}^2 \setminus \{(0,0)\}$, the directional derivative $\nabla_v f(0,0)$ of f at $(0,0)$ in the direction v exists, and determine its value.
3. Deduce that $\partial_1 f(0,0)$ and $\partial_2 f(0,0)$ exist and determine their value.
4. For $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, compute $\partial_1 f(x,y)$ and $\partial_2 f(x,y)$.
5. Are $\partial_1 f$ and $\partial_2 f$ ~~are~~ continuous at $(0,0)$?
6. Is f differentiable at $(0,0)$? If it is the case, determine $d_{(0,0)} f$.

Exercise 4. The three questions of this exercise are independent from each other.

1. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function of class C^1 and define:

$$\begin{array}{rccl} g: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & f(xy, -x, y^2). \end{array}$$

   Compute the first order partial derivatives of g.

2. Let

$$\begin{array}{rccl} f: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & x\sin(y) + y^2. \end{array}$$

   Check that f is of class C^1, and for $(x,y) \in \mathbb{R}^2$, determine $d_{(x,y)} f$.

3. Let

$$\begin{array}{rccl} f: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases} \end{array}$$

   Show that f is not continuous at $(0,0)$ but that $\partial_1 f(0,0)$ and $\partial_2 f(0,0)$ exist (and determine them).