No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

The marks are given as a guide only; the final marking scheme might differ *slightly* from the marks provided here.

Exercises 1 and 2 are common with PCC2.

Exercise 1 (*3 marks*). The two questions of this exercise are independent from each other.

1. Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}^2\to\mathbb{R}$ be two functions of class C^1. Define:

$$\begin{aligned}\varphi:\ \mathbb{R}^2 &\longrightarrow \mathbb{R}^2\\ (x,y) &\longmapsto \Big(f(x+y^2),\, g(2y, x+y^3)\Big).\end{aligned}$$

   Explain why φ is of class C^1 and for $(x,y)\in\mathbb{R}^2$ determine the Jacobian matrix $J_{(x,y)}\varphi$ of φ at (x,y) in terms of the partial derivatives of f and g at appropriate points.

2. Let $u:\mathbb{R}^2\to\mathbb{R}$ be a function.

   a) We assume that:

$$\lim_{t\to 0}\frac{u(t,0)-u(0,0)}{t}=0 \qquad\text{and}\qquad \lim_{t\to 0}\frac{u(0,t)-u(0,0)}{t}=1.$$

   i) Can we deduce that $\partial_1 u(0,0)$ exists? if it is the case, determine the value of $\partial_1 u(0,0)$.
   ii) Can we deduce that $\partial_2 u(0,0)$ exists? if it is the case, determine the value of $\partial_2 u(0,0)$.
   iii) Can we deduce that u is differentiable at $(0,0)$? if it is the case, determine $\mathrm{d}_{(0,0)}u$.

   b) We assume, moreover, that u is of class C^1 on $\mathbb{R}^2$. Determine the directional derivative of u at $(0,0)$ in the direction $v=(1,2)$.

   c) We assume, moreover, that:

$$\lim_{t\to 0}\frac{\partial_1 u(t,0)-\partial_1 u(0,0)}{t}=1,\qquad \lim_{t\to 0}\frac{\partial_1 u(0,t)-\partial_1 u(0,0)}{t}=2,$$

$$\lim_{t\to 0}\frac{\partial_2 u(t,0)-\partial_2 u(0,0)}{t}=3,\qquad \lim_{t\to 0}\frac{\partial_2 u(0,t)-\partial_2 u(0,0)}{t}=4.$$

   Determine the value of $\partial^2_{1,2}u(0,0)=\partial_1(\partial_2 u)(0,0)$.

Exercise 2 (*5.5 marks*). Let $\Omega=\big\{(x,y)\in\mathbb{R}^2 \,\big|\, 0<y<x\big\}$, and define

$$\begin{aligned}\varphi:\ \Omega &\longrightarrow \Omega'\\ (x,y) &\longmapsto (x^2-y^2,\, y),\end{aligned}$$

where $\Omega'\subset\mathbb{R}^2$ is the range of φ (that you will determine in the first question).

1. a) Determine the range $\Omega'=\varphi(\Omega)$ of φ and show that φ is a C^2-diffeomorphism; you will also determine an expression of φ^{-1}.
   We denote by (s,t) the new coordinates defined by φ.

   b) Plot, on two different figures, the sets Ω and Ω'.

2. Find all functions $g:\Omega'\to\mathbb{R}$ of class C^2 that satisfy:

$$\forall (s,t)\in\Omega',\ \partial^2_{2,2}g(s,t)=0.$$

3. Find all functions $f:\Omega\to\mathbb{R}$ of class C^2 that are satisfy the following partial differential equation:

$$(E_1)\qquad xy^2\partial^2_{1,1}f(x,y)+2x^2y\partial^2_{1,2}f(x,y)+x^3\partial^2_{2,2}f(x,y)+(x^2-y^2)\partial_1 f(x,y)=0.$$

Exercise 3 (*3 marks*). Let $\theta : \mathbb{R}^2 \to \mathbb{R}$ be a function of class C^1, and let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable function such that:

$$\forall (x,y) \in \mathbb{R}^2,\ J_{(x,y)}\varphi = \begin{pmatrix} \cos(\theta(x,y)) & -\sin(\theta(x,y)) \\ \sin(\theta(x,y)) & \cos(\theta(x,y)) \end{pmatrix}.$$

We denote by φ_1 and φ_2 the components of φ, i.e., $\varphi_1 : \mathbb{R}^2 \to \mathbb{R}$ and $\varphi_2 : \mathbb{R}^2 \to \mathbb{R}$ are such that

$$\forall (x,y) \in \mathbb{R}^2,\ \varphi(x,y) = \big(\varphi_1(x,y), \varphi_2(x,y)\big).$$

1. For $(x,y) \in \mathbb{R}^2$, determine the differentials $\mathrm{d}_{(x,y)}\varphi_1$ and $\mathrm{d}_{(x,y)}\varphi_2$ of φ_1 and φ_2 at (x,y).
2. Explain why the differential forms $\mathrm{d}\varphi_1$ and $\mathrm{d}\varphi_2$ are closed and of class C^1, and use this fact to deduce that θ is constant.
3. Deduce a simpler form for φ.

Exercise 4 (*3 marks*). Let

$$f : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & x^3 + y^3 + 3xy - 8. \end{array}$$

We assume that there exists an open interval I containing 0 and a function $\varphi : I \to \mathbb{R}$ of class C^∞ such that:

$$(*) \qquad \forall x \in I,\ f\big(x, \varphi(x)\big) = 0.$$

We also denote by C the level set of f at level 0:

$$C = \big\{(x,y) \in \mathbb{R}^2 \,\big|\, f(x,y) = 0\big\}.$$

1. Determine the value of $\varphi(0)$.
2. a) For $x \in I$, differentiate the relation $(*)$ to find a formula for $\varphi'(x)$ in terms of x and $\varphi(x)$ only.
   Note. We assume that I is small enough so that the formula you obtain is well-defined.
   b) Deduce the value of $\varphi'(0)$.
   c) Give a non-nil normal vector to C at $\big(0, \varphi(0)\big)$ and check that it is consistent with the value of $\varphi'(0)$.
3. Determine the value of $\varphi''(0)$.

Exercise 5 (*3 marks*). Let U and V be two open subsets of $\mathbb{R}^2$ and let

$$\varphi : \begin{array}{ccc} U & \longrightarrow & V \\ (x,y) & \longmapsto & \big(\varphi_1(x,y), \varphi_2(x,y)\big) \end{array}$$

be a C^k-diffeomorphism (with $k \geq 1$).
Let $(u_0, v_0) \in V$ and define:

$$C_1 = \big\{(x,y) \in \mathbb{R}^2 \,\big|\, \varphi_1(x,y) = u_0\big\} \quad \text{and} \quad C_2 = \big\{(x,y) \in \mathbb{R}^2 \,\big|\, \varphi_2(x,y) = v_0\big\}.$$

Since φ is a diffeomorphism, it is easy to conclude that the intersection $C_1 \cap C_2$ consists of a unique point $a_0 = (x_0, y_0) \in U$. The goal of this exercise is to show that C_1 and C_2 are not tangent at a_0.

1. Give a non-nil normal vector $\vec{n}_1$ to C_1 at a_0 and a non-nil normal vector $\vec{n}_2$ to C_2 at a_0.
   Justify why the vectors $\vec{n}_1$ and $\vec{n}_2$ you gave are non-nil.
2. Deduce a non-nil tangent vector $\vec{t}_1$ to C_1 at a_0 and a non-nil tangent vector $\vec{t}_2$ to C_2 at a_0.
3. Show that $\vec{t}_1$ and $\vec{t}_2$ are not collinear.

Exercise 6 (*2.5 marks*).

1. Let U be an open subset of $\mathbb{R}^n$, let $f : U \to \mathbb{R}$ be a function of class C^2 and let $a_0 \in U$. Recall the second order Taylor–Young formula for f at a_0 in terms of the Jacobian matrix and the Hessian matrix of f.
2. Let

$$f : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & e^x y + xy. \end{array}$$

Determine the second order Taylor–Young expansion of f at $a_0 = (0,1)$. *You will give the Jacobian matrix and the Hessian matrix of f at a_0.*