INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Solution of Math Test #4

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March 28, 2022

Exercise 1.

1. a) By the Magic Lemma, the radius of convergence of $\sum_n a_n z^n$ is that of $\sum_n 2^n z^n = \sum_n (2z)^n$, namely $R = 1/2$.
   For $x \in (-1/2, 1/2) \setminus \{0\}$:

$$f(x) = \sum_{n=0}^{+\infty} \frac{(2x)^n}{n+1} = \sum_{n=1}^{+\infty} \frac{(2x)^{n-1}}{n} = \frac{1}{2x} \sum_{n=1}^{+\infty} \frac{(2x)^n}{n} = \frac{-\ln(1-2x)}{2x}.$$

   In the case $x = 0$, $f(x) = a_0 = 1$ so that

$$\forall x \in (-1/2, 1/2),\ f(x) = \begin{cases} \dfrac{\ln(1+2x)}{2x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

   b) From the power series expansion of f we have the following second-order Taylor–Young expansion:

$$f(x) \underset{x\to 0}{=} a_0 + a_1 x + a_2 x^2 + o\big(x^2\big) \underset{x\to 0}{=} 1 + x + \frac{4}{3}x^2 + o\big(x^2\big)$$

   from which we conclude that an equation of the tangent line to the graph of f at 0 is

$$y = 1 + x$$

   and since $4/3 > 0$ we conclude that the graph of f lies above Δ in a neighborhood of 0.

2. a) By the Magic Lemma, the radius of convergence of this power series is that of $\sum_n \cos(2\pi n/5) z^n$. Now for $z_0 = 1$, the sequence $\big(\cos(2\pi n/5) z_0^n\big)_n$ is bounded hence, by Abel's Lemma, $R \geq 1$.
   b) Since $n\cos(2\pi n/5) \underset{n\to+\infty}{\not\to} 0$ we conclude that the series $\sum_n n\cos(2\pi n/5)$ diverges, hence $R \leq 1$.
   Finally we conclude that $R = 1$.

3. a) $R = 1$ and

$$\forall x \in (-1, 1),\ f(x) = \frac{1}{1+x} = \sum_{n=0}^{+\infty} (-1)^n x^n.$$

   b) Observe that

$$\forall x \in \mathbb{R} \setminus \{-1\},\ g(x) = 1 - \frac{3}{x+1} = 1 - 3f(x)$$

   hence the radius of convergence of g is that of f, and the domain of convergence of its power series is that of f too. Hence

$$\forall x \in (-1, 1),\ g(x) = \sum_{n=0}^{+\infty} a_n x^n$$

   where

$$\forall n \in \mathbb{N},\ a_n = \begin{cases} -2 & \text{if } n = 0 \\ -3(-1)^n & \text{if } n \neq 0 \end{cases}$$

4. a) We know that

$$\forall t \in \mathbb{R},\ \mathrm{e}^{-t^3} = \sum_{p=0}^{+\infty} \frac{(-t)^3}{p!}$$

   (the radius of convergence of this power series is $R = +\infty$) hence, by term by term integration:

$$\forall x \in \mathbb{R},\ F(x) = \sum_{p=0}^{+\infty} \int_0^x \frac{(-1)^p t^{3p}}{p!}\,\mathrm{d}t = \sum_{p=0}^{+\infty} (-1)^p \frac{x^{3p+1}}{p!(3p+1)}$$

The sequence $(a_n)_{n\in\mathbb{N}}$ to determine is hence:

$$\forall n \in \mathbb{N},\ a_n = \begin{cases} (-1)^p \dfrac{1}{p!(3p+1)} & \text{if } n = 3p+1 \text{ for } p \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

b) The series defining $F(1)$ is an alternating series:

$$F(1) = \sum_{p=0}^{+\infty} (-1)^p \frac{1}{p!(3p+1)}$$

and satisfies the alternating series test. Hence, for all $P \in \mathbb{N}$,

$$\left| \sum_{p=P+1}^{+\infty} (-1)^p \frac{1}{p!(3p+1)} \right| \leq \frac{1}{(P+1)!(3P+4)}$$

so we only need to find P such that

$$\frac{1}{(P+1)!(3P+4)} \leq 10^{-3}$$

i.e.,

$$(P+1)!(3P+4) \geq 1000.$$

If we take $P = 4$ we have:

$$(P+1)!(3P+4) = 5! \times 16 = 120 \times 16 > 1000.$$

We can hence take $N = 3P + 1 = 13$.

Exercise 2.

1. $a_0 = y(0) = 0$ (by the initial condition).
   For $x \in (-R, R)$,

$$y'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1} = \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n,$$

so that

$$xy'(x) = \sum_{n=1}^{+\infty} n a_n x^n = \sum_{n=0}^{+\infty} n a_n x^n$$

and

$$(x-1)y'(x) + y(x) = xy'(x) - y'(x) + y(x) = \sum_{n=0}^{+\infty} \big(n a_n - (n+1) a_{n+1} + a_n \big) x^n = \sum_{n=0}^{+\infty} (n+1)(a_n - a_{n+1}) x^n.$$

The right hand side of the differential equation has the following power series expansion:

$$\forall x \in (-1, 1),\ \frac{1}{x+1} = \sum_{n=0}^{+\infty} (-1)^n x^n$$

hence we must have:

$$\forall x \in (-R, R) \cap (-1, 1),\ \sum_{n=0}^{+\infty} (n+1)(a_n - a_{n+1}) = \sum_{n=0}^{+\infty} (-1)^n x^n$$

and we conclude, by the Identity Theorem, that

$$\forall n \in \mathbb{N},\ (n+1)(a_n - a_{n+1}) = (-1)^n,$$

i.e.,

$$\forall n \in \mathbb{N},\ a_{n+1} - a_n = \frac{(-1)^{n+1}}{n+1}.$$

Hence, using a telescopic sum and the fact that $a_0 = 0$:

$$\forall n \in \mathbb{N}^*,\ a_n = a_0 + \sum_{k=0}^{n-1}(a_{k+1} - a_k) = \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{k+1} = \sum_{k=1}^{n} \frac{(-1)^k}{k}.$$

2. a) We know that

$$\forall x \in (-1, 1],\ \ln(1 + x) = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n} x^n$$

(and that the radius of convergence of the power series is 1) hence:

$$\forall x \in (-1, 1],\ S(x) = -\ln(1 + x).$$

and $r = 1$. This series converges for $x = 1$ (its the alternating harmonic series).

b) $\lim_{n\to+\infty} a_n = S(1) = \ln(2)$.

c) Let $z \in \mathbb{C}^*$. Then

$$\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| = \frac{a_{n+1}}{a_n}|z| \underset{n\to+\infty}{\longrightarrow} \frac{\ln(2)}{\ln(2)}|z| = |z|$$

from which we conclude (by the Ratio Test) that $R = 1$.

3. a) i) $D = (-1, 1) \cup (1, +\infty)$.

ii) Let $x \in D$. Then

$$f'(x) = \frac{1}{(x-1)(x+1)} - \frac{\ln(1 + x)}{(x-1)^2}$$

hence

$$(x - 1)f'(x) + f(x) = \frac{1}{x+1} - \frac{\ln(1 + x)}{x-1} + \frac{\ln(1 + x)}{x-1} = \frac{1}{x+1}.$$

Moreover, $f(0) = 0$. Hence f is solution of Problem (P) on D.

b) Since f admits a power series expansion with a non-nil radius of convergence, we conclude that $R_f = R = 1$ and that

$$\forall x \in (-1, 1),\ f(x) = y(x) = \sum_{n=0}^{+\infty} a_n x^n.$$

Exercise 3.

1. (1) Define

$$\forall n \in \mathbb{N},\ u_n = \frac{2^n}{\sqrt{n!}}.$$

The sequence $(u_n)_{n\in\mathbb{N}}$ is a sequence with positive terms, and we can use the ratio test:

$$\frac{u_{n+1}}{u_n} = \frac{2^{n+1}}{\sqrt{(n+1)!}} \frac{\sqrt{n!}}{2^n} = \frac{2}{\sqrt{n+1}} \underset{n\to+\infty}{\longrightarrow} 0 < 1$$

hence Series (1) converges.

(2) Define

$$\forall n \in \mathbb{N}^*,\ u_n = \ln\left(1 + \frac{\cos n}{n^2}\right).$$

Then, since cos is bounded, we have $\cos(n)/n^2 \underset{n\to+\infty}{\longrightarrow} 0$, and hence

$$|u_n| \underset{n\to+\infty}{\sim} \frac{|\cos n|}{n^2}.$$

Now

$$\forall n \in \mathbb{N}^*,\ 0 \leq \frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$$

and $1/n^2$ is the general term of a convergent series (by Riemann with $\alpha = 2 > 1$) hence, by the Comparison Test, Series (2) converges absolutely, hence converges.

(3) Define
$$\forall n \in \mathbb{N}^*,\ v_n = (-1)^n\left(\sqrt{n+1}-\sqrt{n}\right) = \frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}.$$
We notice that $(v_n)_{n\in\mathbb{N}^*}$ is an alternating sequence, that $\big(|v_n|\big)_{n\in\mathbb{N}^*}$ is decreasing and that $v_n \underset{n\to+\infty}{\longrightarrow} 0$. Hence, by the alternating series test, Series (3) converges.

2. Since $\alpha > 0$ we have $(-1)^n/n^\alpha \underset{n\to+\infty}{\longrightarrow} 0$ and hence:
$$\exp\left(\frac{(-1)^n}{n^\alpha}\right) \underset{n\to+\infty}{=} 1 + \frac{(-1)^n}{n^\alpha} + \frac{1}{2n^{2\alpha}} + \frac{(-1)^n}{3!n^{3\alpha}} + \frac{1}{4!n^{4\alpha}} + o\left(\frac{1}{n^{4\alpha}}\right),$$
hence
$$u_n = \exp\left(\frac{(-1)^n}{n^\alpha}\right) - 1 - \frac{1}{2n^{2\alpha}} \underset{n\to+\infty}{=} \frac{(-1)^n}{n^\alpha} + \frac{(-1)^n}{3!n^{3\alpha}} + \frac{1}{4!n^{4\alpha}} + o\left(\frac{1}{n^{4\alpha}}\right),$$
Define the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ as:
$$\forall n \in \mathbb{N},\ a_n = \frac{(-1)^n}{n^\alpha} + \frac{(-1)^n}{3!n^{3\alpha}}, \qquad b_n = u_n - a_n.$$
Since $\alpha > 0$, $\sum_n (-1)^n/n^\alpha$ and $\sum_n (-1)^n/n^{3\alpha}$ are convergent series (alternating Riemann series), hence $\sum_n a_n$ converges. Now
$$b_n \underset{n\to+\infty}{=} \frac{1}{4!n^{4\alpha}} + o\left(\frac{1}{n^{4\alpha}}\right) \underset{n\to+\infty}{\sim} \frac{1}{4!n^{4\alpha}} > 0,$$
hence, by the equivalent test, $\sum_n b_n$ converges if and only if $4\alpha > 1$, i.e., if and and only if $\alpha > 1/4$. We hence conclude that Series (3) converges if and only if $\alpha > 1/4$.

Exercise 4.

1.
$$\frac{1}{1+n^3} \underset{n\to+\infty}{\sim} \frac{1}{n^3} > 0$$
hence, by the equivalent test, the series defining S converges, hence S is well-defined.

Since the general term of the series defining S is positive, the sequence $(S_N)_{N\in\mathbb{N}}$ is increasing, hence $S - S_N > 0$.

We now use the Integral Comparison Test: the function
$$f : \mathbb{R}_+ \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{1}{1+x^3}$$
is non-negative and decreasing hence, for $N \in \mathbb{N}$,
$$S - S_N = \sum_{n=N+1}^{+\infty} \frac{1}{1+n^3} \leq \int_N^{+\infty} f(x)\,\mathrm{d}x \leq \int_N^{+\infty} \frac{\mathrm{d}x}{x^3} = \frac{1}{2N^2}.$$

2. We hence have:
$$S_{1000} \leq S \leq S_{1000} + \frac{1}{2\cdot 1000^2} = S_{1000} + 5\cdot 10^{-7}.$$
From the value for S_{1000} given, we conclude that
$$1.6865028 < S_{1000} < 1.6865029$$
so that
$$1.6865028 < S_{1000} \leq S \leq 1.6865029 + 5\cdot 10^{-7} = 1.6865034$$
so that
$$S = 1.68650\ldots$$
(and the next digit is either 2 or 3).