SCAN 2 – Math Test #4 – 2h

March 28, 2022

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

The marks are given as a guide only; the final marking scheme might differ *slightly* from the marks provided here.

Exercises 1 and 2 are common with PCC2.

Exercise 1 (*8.5 marks*). The four questions of this exercise are independent of each other.

1. Define the sequence $(a_n)_{n\in\mathbb{N}}$ as:

$$\forall n \in \mathbb{N},\ a_n = \frac{2^n}{n+1}.$$

   a) Determine the radius of convergence R of the power series f defined by $f(x) = \sum_{n=0}^{+\infty} a_n x^n$, and express f in terms of usual functions on $(-R, R)$.

   b) Specify the graph of f in a neighborhood of 0 without studying the function f explicitly (i.e., an equation of the tangent line to the graph of f at 0 and the relative position of the graph of f with respect to this tangent line).

2. Let R be the radius of convergence of the power series $\sum_n n\cos\left(\frac{2\pi n}{5}\right) z^n$.

   b) Show that $R \geq 1$.

   c) What is the nature of the numerical series $\sum_n n\cos\left(\frac{2\pi n}{5}\right)$? what can you deduce about R?

3. a) Recall the power series expansion of the function $f : x \mapsto \frac{1}{x+1}$ as well as its radius of convergence R.

   b) Deduce the power series expansion of the function $g : x \mapsto \frac{x-2}{x+1}$ as well as the interval of convergence; your answer should be of the form

$$(*) \qquad \forall x \in I,\ g(x) = \sum_{n=0}^{+\infty} a_n x^n$$

   where I is the real interval of convergence of the power series (closed, open or semi-open) you will determine, and $(a_n)_{n\in\mathbb{N}}$ is the sequence of coefficients you will determine.

4. Let

$$F : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \int_0^x e^{-t^3}\, dt.$$

   a) Determine the power series expansion of the function F and specify the interval of convergence. You will give the answer in a form similar to $(*)$ above.

   b) Determine a value of $N \in \mathbb{N}$ such that the partial sum $\sum_{n=0}^{N} a_n$ is an approximation of $F(1)$ with error less than 10^{-3}.

Exercise 2 (*6 marks*). Let (P) be the following initial value problem:

$$(\mathrm{P}) \qquad \begin{cases} (x-1)y'(x) + y(x) = \dfrac{1}{x+1} \\ y(0) = 0 \end{cases}$$

We assume that there exists a power series $y(x) = \sum_{n=0}^{+\infty} a_n x^n$ of radius of convergence $R > 0$ which is a solution of Problem (P).

1. Explain why $a_0 = 0$ and show that

$$\forall n \in \mathbb{N}^*,\ a_n = \sum_{k=1}^{n} \frac{(-1)^k}{k}.$$

2. a) Give, without any justifications, the radius of convergence r and the sum $S(x)$ of the power series

$$S(x) = \sum_{k=1}^{+\infty} \frac{(-1)^k}{k} x^k.$$

   Does the series converge for $x = r$?

   b) We assume that the function S is continuous at $x = r$. Determine the value of $\lim_{n \to +\infty} a_n$.

   c) Determine the value of the radius of convergence R of the solution y of Problem (P).

3. a) i) Determine the largest subset D of $\mathbb{R}$ such that for all $x \in D$ the expression

$$f(x) = \frac{\ln(x+1)}{x-1}$$

   is defined. Let $f : D \to \mathbb{R}$ be the corresponding function.

   ii) Check that f is a solution of Problem (P) on D.

   b) We admit that f possesses a power series expansion with radius of convergence $R_f > 0$. Deduce from the previous questions the coefficients of the power series expansion of f as well as the the value of R_f.

Exercise 3 (*3 marks*). The questions of this exercise are independent from each other.

1. Determine the convergence of the following series:

$$(1)\ \sum_n \frac{2^n}{\sqrt{n!}} \qquad (2)\ \sum_n \ln\left(1 + \frac{\cos n}{n^2}\right) \qquad (3)\ \sum_n (-1)^n\left(\sqrt{n+1} - \sqrt{n}\right)$$

2. For what values of $\alpha \in \mathbb{R}_+^*$ does the following numerical series converge?

$$\sum_n \left(\exp\left(\frac{(-1)^n}{n^\alpha}\right) - 1 - \frac{1}{2n^{2\alpha}}\right).$$

Exercise 4 (*2.5 marks*). The goal of this exercise is to determine a numerical approximation of the following sum:

$$S = \sum_{n=0}^{+\infty} \frac{1}{1+n^3},$$

by using the sequence of partial sums $(S_N)_{N \in \mathbb{N}}$ defined by:

$$\forall N \in \mathbb{N},\ S_N = \sum_{n=0}^{N} \frac{1}{1+n^3}.$$

1. Show that S is well-defined and that there exists a decreasing sequence $(\alpha_N)_{N \in \mathbb{N}^*}$ (that you will explicitly determine) such that

$$\forall N \in \mathbb{N}^*,\ 0 \leq S - S_N \leq \alpha_N$$

   and such that $\alpha_N \underset{N \to +\infty}{\longrightarrow} 0$.

2. You're given

   $S_{1000} = 1.68650284283837408554843833721958917688068675429394958672517493272222163351092838481576788 4\ldots$

   (where all the digits displayed are exact digits).

   Deduce an approximation of S correct to as many decimal places as possible.