INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Math Test #5 — 2h

June 15, 2022

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

The marks are given as a guide only; the final marking scheme might differ *slightly* from the marks provided here.

Exercise 1 (*6 marks*).

1. Let

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

   a) Explain why the matrix A is diagonalizable, and diagonalize A in an orthonormal basis.

   b) Determine the expression of the quadratic form q on $\mathbb{R}^3$ the matrix of which, in the standard basis of $\mathbb{R}^3$ is A. Determine the signature and the rank of q.

2. Let $a, b, c \in \mathbb{R}$ and define:

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & a & c \\ 0 & c & b \end{pmatrix}.$$

   Without computing the characteristic polynomial of B:

   a) Determine an eigenvalue and an eigenvector of B.

   b) Under what conditions on a, b, c is the matrix B the matrix of an inner product on $\mathbb{R}^3$?

3. Let C be a real symmetric matrix with all its eigenvalues of multiplicity 1. Explain why the column vectors $U = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $V = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ can't simultaneously be eigenvectors of C.

Exercise 2 (*6.5 marks*). Let f be the function defined by

$$f : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x, y) & \longmapsto & (x^2 + y^2)e^{-x}. \end{array}$$

1. Determine the critical points of f and study their nature.

2. Define

$$\Delta = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 9,\ x \geq 0\}.$$

   a) Plot Δ.

   b) Justify that f admits a global minimum and a global maximum on Δ.

   c) Determine the global minimum and the global maximum of f on Δ, and specify where they are attained.

3. Does f posses a global minimum on $\mathbb{R}^2$? does f possess a global maximum on $\mathbb{R}^2$?

Exercise 3 (*7.5 marks*). We recall the following theorem:

Theorem. *Let $(E, \langle\cdot,\cdot\rangle)$ be a pre-Hilbert space with associated norm $\|\cdot\|$. Let F be a finite-dimensional subspace of E, and denote by p_F the orthogonal projection of E on F. Then, for all $x \in E$ one has:*

$$\|x - p_F(x)\| = \inf_{y \in F} \|x - y\|.$$

1. a) Illustrate that theorem using a figure.

   b) Prove that theorem.

   Let $c \in \mathbb{R}$. For $a, b \in \mathbb{R}$ we define:

$$I(a,b) = \int_0^{\pi} \Big(c - \big(a\cos(x) + b\sin(x)\big)\Big)^2 \, dx.$$

From now on, the goal is to determine

$$m = \inf_{(a,b)\in\mathbb{R}^2} I(a,b).$$

Let $E = C([0,\pi])$ be the vector space of real-valued continuous functions on $[0,\pi]$. We recall that the standard dot product on E is given by:

$$\forall f, g \in E,\ \langle f, g\rangle = \int_0^{\pi} f(x)g(x)\,dx.$$

Let $F = \mathrm{Span}\{u, v\}$ be the subspace of E generated by the functions

$$u : \begin{array}{rcl} [0,\pi] & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \cos(x) \end{array} \qquad\qquad v : \begin{array}{rcl} [0,\pi] & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \sin(x). \end{array}$$

You may use, without any justifications, the following trigonometric formulas:

$$\int_0^{\pi} \cos^2(x)\,dx = \frac{\pi}{2}. = \int_0^{\pi} \sin^2(x)\,dx = \frac{\pi}{2}.$$

2. Use the Gram-Schmidt process to determine an orthonormal (with respect to $\langle\cdot,\cdot\rangle$) basis of F.

3. Explain why $I(a,b)$ represents the distance (with respect to $\langle\cdot,\cdot\rangle$) from $\hat{c}$ to F, where $\hat{c}$ is the element of E defined by:

$$\hat{c} : \begin{array}{rcl} [0,\pi] & \longrightarrow & \mathbb{R} \\ x & \longmapsto & c. \end{array}$$

4. We are going to compute m using two different methods (and Question 4a and 4b are hence independent).

   a) Determine the orthogonal projection of $\hat{c}$ on F, and deduce the value of m.

   b) i) Let $a, b \in \mathbb{R}$ such that $(\hat{c} - (au + bv)) \in F^{\perp}$. Without using the expression of the orthogonal projection, determine a and b in terms of c.

      ii) Recover the result of Question 4a.