Written exam number 1, 23th of October, 2023

Mathematics, SCAN 2$^{\text{nd}}$ year, 2023–2024

Duration: 1:30.
No document allowed, no calculation tool allowed.
Exercises are independent and can be treated in any order.

Exercise 1. ***Nature of some improper integrals (∼ 4pts)***
Determine the nature (convergence) of the following improper integrals.

1. $\displaystyle\int_1^{+\infty} \frac{1}{\sqrt{t}} \ln\left(1+\frac{1}{t}\right) dt$ 2. $\displaystyle\int_1^{+\infty} \frac{2+\sin(t)}{\sqrt{t}} dt$ 3. $\displaystyle\int_0^{+\infty} \frac{\cos(t)}{\sqrt{e^t-1}} dt$

Exercise 2. ***A sequence of improper integrals (∼ 5pts)***
For n in $\mathbb{N}$, let us consider the integral $I_n = \displaystyle\int_0^1 \frac{(\ln t)^n}{\sqrt{t}} dt$.

1. Determine the nature (convergence) of I_0, and then of I_n for every n in $\mathbb{N}$.
2. Calculate the value of I_0.
3. Using an integration by parts, find (for $n \geq 1$) a relation between I_n and I_{n-1} (an expression of I_n in terms of I_{n-1}).
4. Deduce from this relation the value of I_n.

Exercise 3. ***Numerical series (∼ 5pts)***
1. Let $\sum_n u_n$ and $\sum_n v_n$ denote series of real numbers. Qualify each of the following statements as true or false, and, if false, provide a counterexample (no additional comment is required, just "True" if true or a counterexample if false).

a) $u_n \underset{n\to+\infty}{\sim} v_n$ and $\sum_n v_n$ converges $\Longrightarrow \sum_n u_n$ converges.

b) $u_n = \underset{n\to+\infty}{o}(v_n)$ and $\sum_n v_n$ converges $\Longrightarrow \sum_n u_n$ converges.

c) $u_n \geq 0$ for all n and $\sum_n u_n$ converges $\Longrightarrow \sum_n u_{2n}$ converges.

2. For the (converging) series $\displaystyle\sum_{k\geq 1} \frac{(-1)^k}{k^2}$, find n large enough so that the remainder associated with the partial sum S_n be not larger than 0.01 (justify).

3. Only one among the following two statements is true; provide a counterexample to the wrong statement (drawing with explanations welcomed).

1. If a series $\sum_n u_n$ converges, then $u_n \to 0$ as $n \to +\infty$.
2. If the improper integral $\displaystyle\int_0^{+\infty} f(x)\,dx$ converges (f being a *continuous* function $[0,+\infty) \to \mathbb{R}$), then $f(x) \to 0$ as $x \to +\infty$.

Exercise 4. ***Series-integral comparison (∼ 6.5pts)***
Let us consider a function $f : [1, +\infty) \to \mathbb{R}$ which is, on $[1, +\infty)$,

- piecewise continuous,
- and positive,
- and non-increasing.

1. a) Let k and n denote integers with $k \geq 1$ and $n \geq 2$. Using the sub-figures of figure 1, provide lower and upper bounds for the integrals:

$$\int_k^{k+1} f(x)\,dx \quad \text{and} \quad \int_1^n f(x)\,dx$$

in terms of general terms or partial sums of the series $\sum_{k\geq 1} f(k)$.

b) Which statement (from the course) can you deduce from these bounds? (justify).

2. For k in $\mathbb{N}^*$, let us consider the quantity: $\delta_k = f(k) - \int_k^{k+1} f(x)\,dx$.

a) On figure 1a, identify (directly on the figure) the domain with area δ_k.

b) On figure 1c, identify (directly on the figure) the domain with area $\sum_{k=1}^{n-1} \delta_k$.

c) Prove the inequality

$$\sum_{k=1}^{n-1} \delta_k \leq f(1) - f(n)\,, \tag{1}$$

and explain how this inequality can be visualized on figure 1c.

d) Deduce from inequality (1) that the series $\sum_{k\geq 1} \delta_k$ converges.

3. Let us consider the quantity Δ defined as the sum of the series $\sum_{k\geq 1} \delta_k$, and let us denote by $(R_n)_{n\geq 1}$ the sequence of its remainders:

$$\Delta = \sum_{k=1}^{+\infty} \delta_k \quad \text{and} \quad R_n = \sum_{k=n+1}^{+\infty} \delta_k\,.$$

a) Prove that

$$\sum_{k=1}^{n-1} f(k) = \int_1^n f(x)\,dx + \Delta - R_{n-1}\,. \tag{2}$$

b) Deduce from this equality an asymptotic expansion for the partial sum of the harmonic series.

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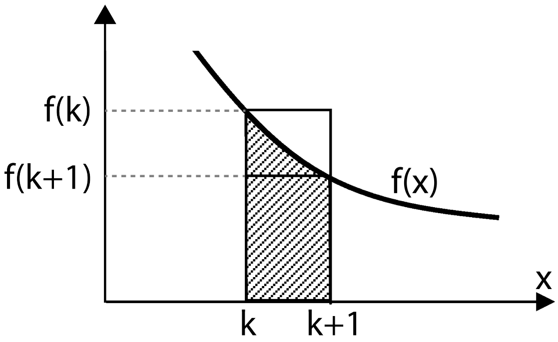
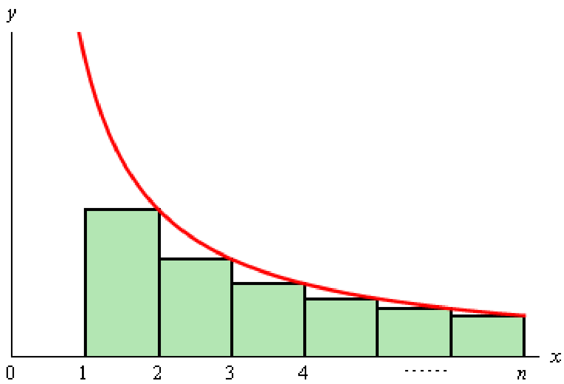


(a)



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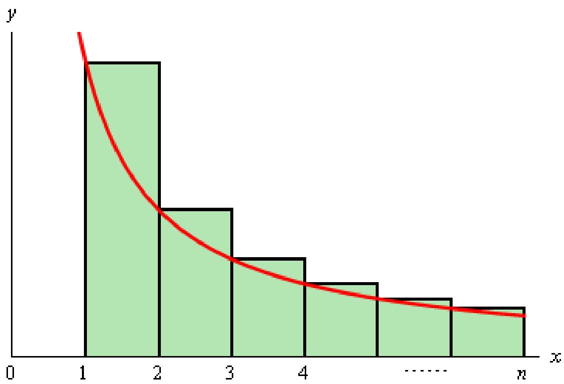

(c)

Figure 1

Short answers for exercise 1

1. The integral is improper at $+\infty$. Since

$$\frac{1}{\sqrt{t}} \ln\left(1+\frac{1}{t}\right) \underset{t\to+\infty}{\sim} \frac{1}{t^{3/2}}\,,$$

it follows from the equivalent test (Riemann test for integrals) that the integral converges.

2. The integral is improper at $+\infty$. Since, for every t in $[1,+\infty)$,

$$\frac{1}{\sqrt{t}} \leq \frac{2+\sin(t)}{\sqrt{t}}\,,$$

it follows from the comparison test (Riemann test for integrals) that the integral diverges.

3. The integral is improper at 0 and at $+\infty$. At 0, since

$$\frac{\cos(t)}{\sqrt{e^t-1}} \underset{t\to 0^+}{\sim} \frac{1}{\sqrt{t}}\,,$$

it follows from the equivalent test (Riemann test for integrals) that the integral converges (at 0); at $+\infty$,

$$\left|\frac{\cos(t)}{\sqrt{e^t-1}}\right| \leq \frac{1}{\sqrt{e^t-1}}\,,$$

and

$$\frac{1}{\sqrt{e^t-1}} \underset{t\to+\infty}{\sim} \frac{1}{\sqrt{e^t}} = e^{-t/2}\,,$$

and

$$e^{-t/2} = \underset{t\to+\infty}{o}\left(\frac{1}{t^2}\right)\,,$$

it follows from the comparison test that the integral is absolutely convergent, thus convergent (at $+\infty$). Conclusion: the integral is convergent at 0 and at $+\infty$, it is therefore convergent.

Short answers for exercise 2

1. The integral I_n (be it for $n=0$ or for $n>0$) is improper at 0. The integral I_0 is convergence (Riemann reference function). Concerning I_n, for every quantity α in $(1/2,1)$,

$$\left|\frac{(\ln t)^n}{\sqrt{t}}\right| = \underset{t\to 0^+}{o}\left(\frac{1}{t^\alpha}\right)\,,$$

and it follows from this equality that the integral is convergent (little oh test, comparison to Riemann reference integral). This equality holds because, for every t in $(0,+\infty)$,

$$\left|t^\alpha \frac{(\ln t)^n}{\sqrt{t}}\right| = t^{\alpha-1/2}(\ln t)^n\,,$$

and since $\alpha - 1/2 > 0$, this last expression goes to 0 as $t \to 0^+$.

2.

$$\begin{aligned} I_0 &= \lim_{x\to 0^+} \int_x^1 \frac{1}{\sqrt{t}}\,dt \\ &= \lim_{x\to 0^+} \left[2\sqrt{t}\right]_x^1 \\ &= \lim_{x\to 0^+} 2-2\sqrt{x} \\ &= 2\,. \end{aligned}$$

3. Integration by parts: for every n in $\mathbb{N}^*$ and for every x in $(0,1]$,

$$\int_x^1 \frac{(\ln t)^n}{\sqrt{t}}\, dt = \Big[2\sqrt{t}(\ln t)^n\Big]_x^1 - \int_x^1 2\sqrt{t} n \frac{(\ln t)^{n-1}}{t}\, dt$$
$$= -2\sqrt{x}(\ln x)^n - 2n \int_x^1 \frac{(\ln t)^{n-1}}{\sqrt{t}}\, dt\,,$$

so that, passing to the limit as $x \to 0^+$,

$$I_n = -2nI_{n-1}\,.$$

4. According to this relation, for every n in $\mathbb{N}$,

$$I_n = (-2)^n n! I_0 = 2(-2)^n n! = (-1)^n 2^{n+1} n!\,.$$

Short answers for exercise 3

1. a) False: $u_n = \dfrac{(-1)^n}{\sqrt{n}} + \dfrac{1}{n}$ and $v_n = \dfrac{(-1)^n}{\sqrt{n}}$.

   b) False: $u_n = \dfrac{1}{n}$ and $v_n = \dfrac{(-1)^n}{\sqrt{n}}$.

   c) True.

2. The alternating series test holds for this series. It follows that, for every n in $\mathbb{N}^*$,

$$|R_n| \leq |u_{n+1}| = \frac{1}{(n+1)^2}\,,$$

and

$$\frac{1}{(n+1)^2} \leq 0.01 \iff n+1 \geq 10 \iff n \geq 9\,.$$

Thus $n = 9$ is large enough to ensure the intended upper bound on the remainder.

3. Statement 1 is true (n-th term test). Statement 2 is false, for instance choose f as the piecewise affine continuous function defined by: f vanishes on $[0,1]$ and, for every positive integer n,

$$f(x) \begin{cases} \text{equals } 0 & \text{if} \quad n \leq x \leq n+1-\dfrac{1}{n^2}\,, \\ \text{increases with slope } n^2 & \text{if} \quad n+1-\dfrac{1}{n^2} \leq x \leq n+1-\dfrac{1}{2n^2}\,, \\ \text{decreases with slope } n^2 & \text{if} \quad n+1-\dfrac{1}{2n^2} \leq x \leq n+1\,. \end{cases}$$

According to this definition, for every n in $\mathbb{N}$,

$$f\left(n+1-\frac{1}{2n^2}\right) = \frac{1}{2}\,,$$

so that $f(x) \not\to 0$ as $x \to +\infty$. On the other hand, still according to this definition,

$$\int_n^{n+1} f(x)\, dx = \frac{1}{4n^2}\,,$$

and since the series $\displaystyle\sum_{n\geq 1} \frac{1}{4n^2}$ converges, it follows that the improper integral $\displaystyle\int_0^{+\infty} f(x)\, dx$ converges.

Short answers for exercise 4

1.

a) According to figure 1a (or integrating the inequality $f(k+1) \leq f(x) \leq f(k)$ on interval $[k, k+1]$),

$$f(k+1) \leq \int_k^{k+1} f(x)\,dx \leq f(k) \tag{3}$$

and summing this inequality between $k = 1$ and $k = n-1$ (or according to figure 1c)

$$\sum_{k=1}^{n-1} f(k+1) \leq \int_1^n f(x)\,dx \leq \sum_{k=1}^{n-1} f(k)\,,$$

or equivalently, in terms of the partial sums S_n of the series $\sum_{k \geq 1} f(k)$,

$$S_n - f(1) \leq \int_1^n f(x)\,dx \leq S_{n-1}\,. \tag{4}$$

b) The left inequality shows that, if the improper integral $\int_1^{+\infty} f(x)\,dx$ converges, then the series $\sum_{k \geq 1} f(k)$ converges. The right inequality shows that, if the series $\sum_{k \geq 1} f(k)$ converges, then the improper integral $\int_1^{+\infty} f(x)\,dx$ converges. In short, the improper integral and the series are of the same nature.

2. a) Small "triangle-like" domains between the graph and the horizontal line at height $f(k)$, for abscissa between k and $k+1$.

b) The union of all these "triangle-like" domains, for abscissa between 1 and n.

c) According to inequalities (3),

$$0 \leq \delta_k \leq f(k) - f(k+1)\,,$$

so that, summing these inequalities for k in $\{1, \ldots, n-1\}$,

$$0 \leq \sum_{k=1}^{n-1} \delta_k \leq f(1) - f(n)\,,$$

and this last inequality is nothing but the intended inequality (1). These two steps show on figure 1c.

d) It follows from inequality (1) that

$$\sum_{k=1}^{n-1} \delta_k \leq f(1)\,,$$

and since the terms δ_k are nonnegative, it follows that $\sum_k \delta_k$ converges.

3. a)

$$\begin{aligned}\sum_{k=1}^{n-1} f(k) &= \sum_{k=1}^{n-1} \left(f(k) - \int_k^{k+1} f(x)\,dx + \int_k^{k+1} f(x)\,dx \right) \\ &= \sum_{k=1}^{n-1} \delta_k + \int_1^n f(x)\,dx\,,\end{aligned}$$

and since (according to the definition of the remainder)

$$\Delta = \sum_{k=1}^{n-1} \delta_k + R_{n-1}\,,$$

the intended equality (2) follows.

b) If f is the function $x \mapsto 1/x$, then equality (2) reads:

$$\sum_{k=1}^{n-1} \frac{1}{k} = \ln(n) + \Delta - R_{n-1}\,,$$

or in other words, since the remainder R_{n-1} goes to 0 as n goes to $+\infty$,

$$\sum_{k=1}^{n-1} \frac{1}{k} = \ln(n) + \Delta + \underset{n\to+\infty}{o}(1)\,,$$

or equivalently,

$$\sum_{k=1}^{n} \frac{1}{k} = \ln(n) + \Delta + \underset{n\to+\infty}{o}(1)\,.$$

Of course, the quantity Δ has a specific value for each function f. For $f(x) = 1/x$ (the harmonic series), this quantity is usually written γ (Euler–Mascheroni constant, https://en.wikipedia.org/wiki/Harmonic_number).