Written exam number 2, 4th of December, 2023

Mathematics, SCAN 2nd year, 2023–2024

Duration: 1:30.
No document nor calculation tool allowed.
The six exercises are independent and can be treated in any order. They are ordered according to the order of the chapters, **not by increasing level of difficulty** (have a look at the two last exercises 5 and 6 on page 4).
Vectors of $\mathbb{R}^2$ are sometimes written (for simplicity and to save space) as rows instead of columns.

Exercise 1. ***Linearization of a function*** $\mathbb{R}^2 \to \mathbb{R}$ ***(∼3.5 pts)***
In this exercise **only the calculations** (without further justifications) are required. We consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined as:

$$f(x,y) = \frac{x^3}{3} - x + \frac{y^2}{2}\,.$$

1. a) Calculate the partial derivatives $\partial_x f$ and $\partial_y f$, and give the expression of the differential $df_{(a,b)}(h,k)$ at some point (a,b) of $\mathbb{R}^2$.
   b) Write the first order Taylor expansion of f at (a,b).
2. The level sets of f are shown on figure 1. On this figure, draw:
   a) the gradient vector $\nabla f(0,1)$ (attached at the point $(0,1)$),
   b) the level set of $df_{(0,1)}$ passing through $(0,1)$, and a few other close level sets of $df_{(0,1)}$.
3. Give the equations of:
   a) the kernel of $df_{(0,1)}$,
   b) the tangent at $(0,1)$ to the level set of f passing through $(0,1)$,
   c) the tangent plane at $\big(0,1,f(0,1)\big)$ to the graph of f and the normal vector to this tangent plane,
   d) the slope of this tangent plane in the direction of steepest ascent.

Exercise 2. ***Nonlinear map*** $\mathbb{R}^2 \to \mathbb{R}^2$ ***(~6.5 pts)***
We consider the (nonlinear) map $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined as:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ xy \end{pmatrix} .$$

1. Calculate the matrix of the differential $DF_{(x,y)}$ and its determinant.
2. Let us consider the set

$$\mathcal{F} = \left\{(x,y) \in \mathbb{R}^2 : \det(DF_{(x,y)}) = 0 \right\}.$$

a) Figure 2a shows the level sets of $x+y$ (in red) and of xy (in blue) in departure space. On this figure, draw the set $\mathcal{F}$ (after providing its equation).

b) On the same figure 2a, indicate in which areas the quantity $\det(DF_{(x,y)})$ is positive / negative.

c) On the same figure 2a, draw the line passing through $(1,1)$ and parallel to $\ker(DF_{(1,1)})$. What do you observe? Why is this consistent?

3. Figure 2b shows the image by F of the canonical grid (of departure space), in arrival space (the coordinates in this arrival space are denoted by (u,v)).

a) Prove that, for every (x,y) in $\mathbb{R}^2$, if $(u,v) = F(x,y)$ then $v \le u^2/4$. For which values of (x,y) is this inequality an equality?

b) Provide an equation for the set $F(\mathcal{F})$ and draw this set on figure 2b.

c) On the same figure 2b, draw the line passing through $(2,1) = F(1,1)$ and parallel to $\mathrm{im}(DF_{(1,1)})$. What do you observe? Why is this consistent?

4. Let us consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} , \quad \text{which is the matrix of} \quad DF_{(1,0)} .$$

a) Figures 3a and 3b show two "canonical grids" in the departure space (x,y) and the arrival space (u,v). On the blank figure 4, draw (in the arrival frame (u,v)) the image by A of the canonical grid of figure 3a, and (in the departure frame (x,y)) the reciprocal image of the canonical grid of figure 3b.

b) Explain how these drawings (on figure 4) are related to figures 2a and 2b.

Exercise 3. ***Linear transformations of*** $\mathbb{R}^2$ ***(~3 pts)***
Each of the figures 5b to 5e represents the image of figure 5a by a linear transformation S of $\mathbb{R}^2$. For each of these transformations, proceed as follows.

- Give the matrix (in the canonical basis of $\mathbb{R}^2$) of the transformation.
- If the transformation is conformal (composition of a homothety and a rotation), provide the ratio of the homothety and the angle of the rotation.
- If the transformation is a projection, provide the directions "onto" and "parallel to" associated with the projection.
- If the transformation is a symmetry, provide the directions "with respect to" and "parallel to" associated with the symmetry.

- If the transformation is nilpotent (meaning: applying it twice maps every vector of $\mathbb{R}^2$ to $(0,0)$), say so.

No justification required.

Exercise 4. ***Conjugation of a*** 2×2 ***matrix (~4 pts)***
Let $(\mathbf{i}, \mathbf{j})$ denote the canonical basis of $\mathbb{R}^2$ and let C denote the square defined by these two vectors (figure 6). We consider the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1/2 & 3/2 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 4 & 2 \\ -2 & 2 \end{pmatrix}.$$

1. Calculate the vectors $AP\mathbf{i}$ and $AP\mathbf{j}$. What is special with these two vectors?
2. On figure 6, draw (in the (x, y) frame, to the left):
   - the vectors $P\mathbf{i}$ and $P\mathbf{j}$,
   - their images $AP\mathbf{i}$ and $AP\mathbf{j}$ by A,
   - the parallelogram $P(C)$ which is the image of C by P.
3. Let us consider the matrix D defined as $D = P^{-1}AP$. Explain why $D = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$ (the calculation of P^{-1} and of the product $P^{-1}AP$ is not required).
4. On figure 6, draw:
   - the sets $D(C)$ and $D^2(C)$ (in the (u, v) frame, to the right),
   - and the sets $A\big(P(C)\big)$ and $A^2\big(P(C)\big)$ (in the (x, y) frame, to the left).
5. For which vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ of $\mathbb{R}^2$ do we have:

   a) $A^n \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $n \to +\infty$? b) $A^n \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $n \to -\infty$?

(no justification required).

Exercise 5. ***Determinant, calculation (∼2 pts)***

1. Calculate the determinant of the matrix $A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & \alpha \\ -1 & 0 & 1 \end{pmatrix}$, where α denotes a real number.

Provide the detail of the calculation.

2. For which values of α is the matrix A invertible? What is its rank when it is not invertible?

Exercise 6. ***Determinant, general properties (∼1.5 pts)***

Let A denote an $n \times n$ (real or complex) invertible matrix (n is an integer not smaller than 2). In each of the following cases, provide the value of $\det(A')$ (in terms of $\det(A)$), without any justification.

1. $A' = 2A$,
2. $A' = A^2$,
3. $A' = A^{-1}$,
4. $A' = A^T$ (transpose of A),
5. $A' = P^{-1}AP$ (where P is an invertible $n \times n$ matrix),
6. A' is derived from A by performing the operation $C_1 \leftarrow C_1 + 2C_2$ on the columns of A,
7. A' is derived from A by performing the operation $C_1 \leftarrow 2C_1 + C_2$ on the columns of A,
8. A' is derived from A by exchanging columns C_1 and C_2 of A.

First name: Last name:

If you need another print of this “figures” part (to change your answer after some mistake), feel free to ask!

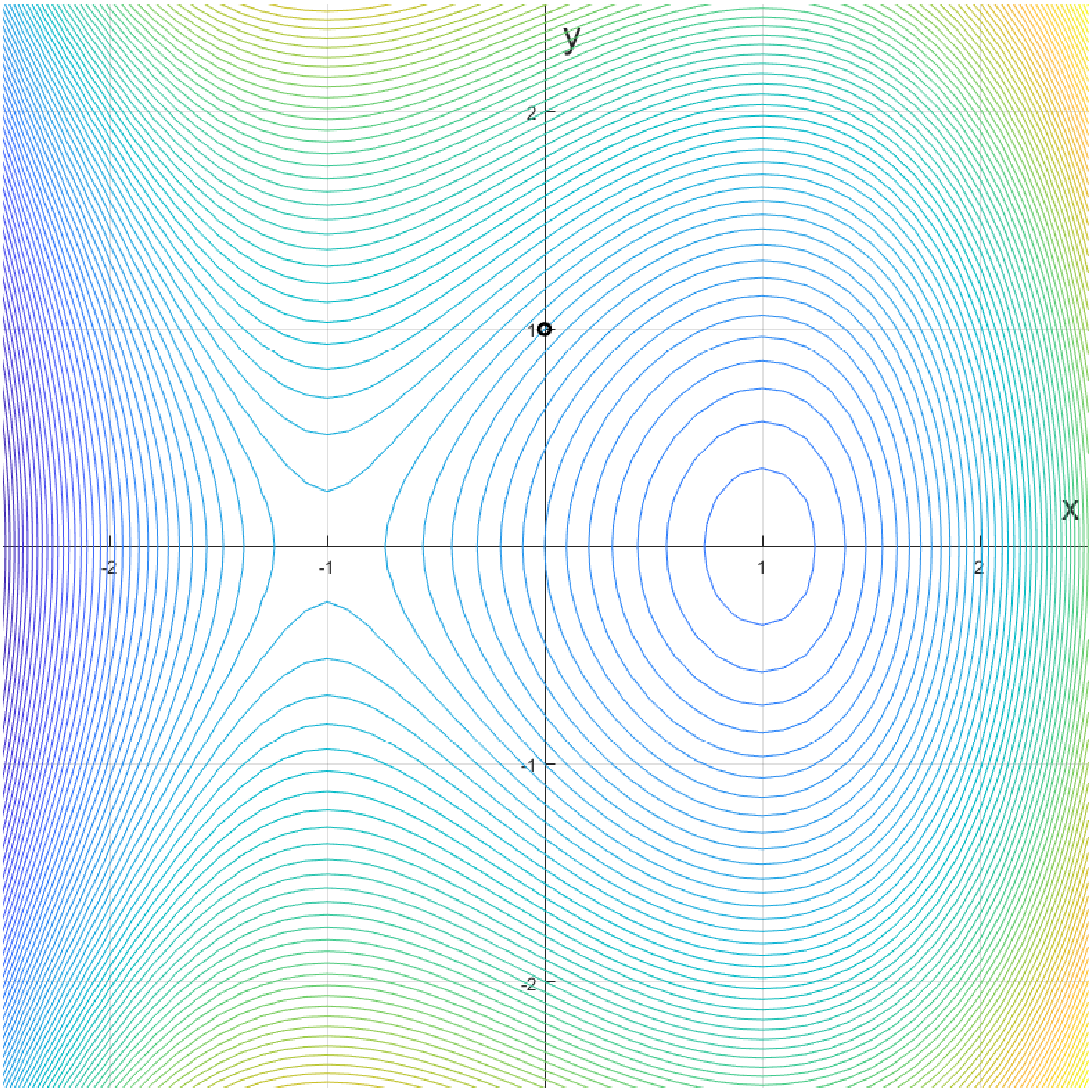
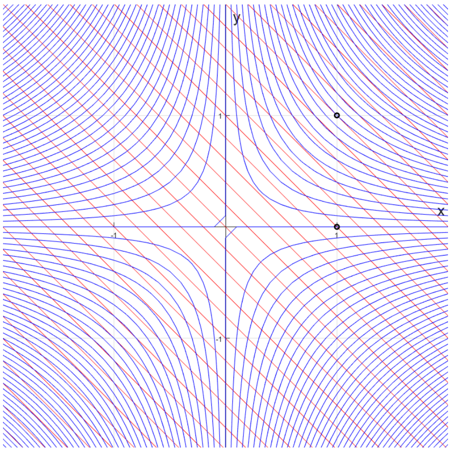
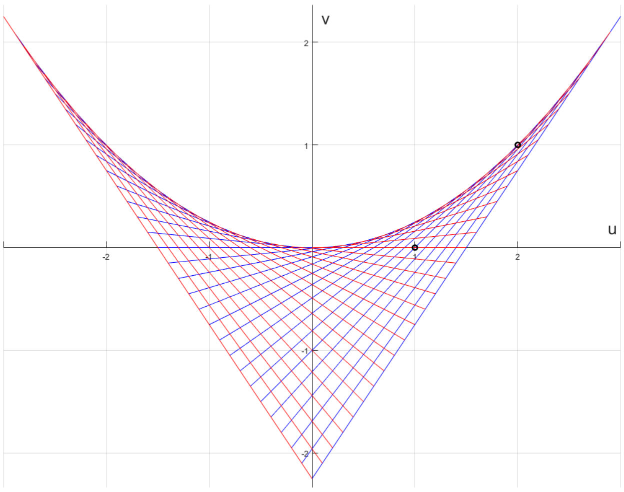


Figure 1: Figure for Exercise 1.

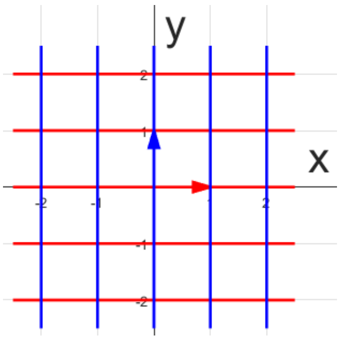


(a) Departure space, level sets of $x + y$ (in red) and xy (in blue).

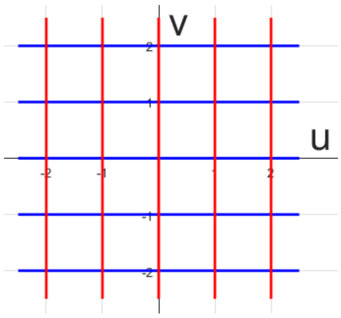


(b) Arrival space, image by F of a grid similar to the one of figure 3a (same colours and directions, number of lines and spacing differ).

Figure 2



(a)



(b)

Figure 3: Canonical grid of $\mathbb{R}^2$ in the departure space (left) and arrival space (right). The fact that the colours of the horizontal and vertical lines are reversed between the two figures is deliberate.

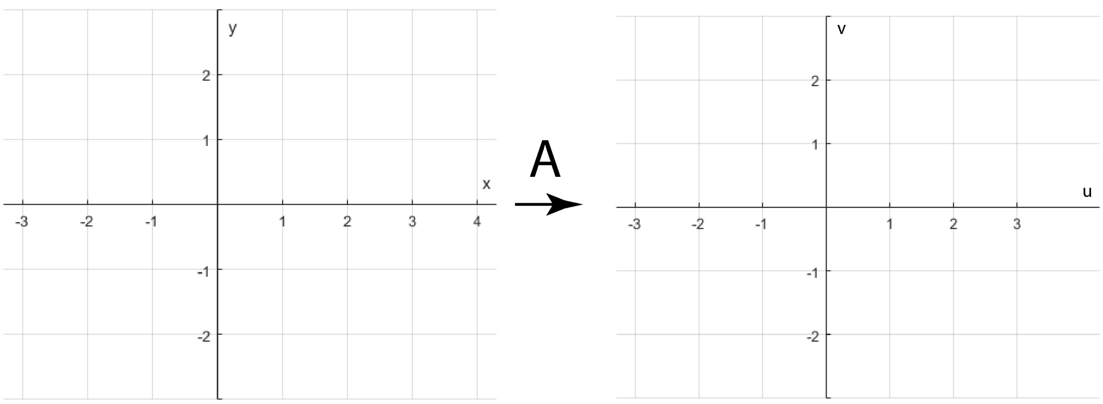
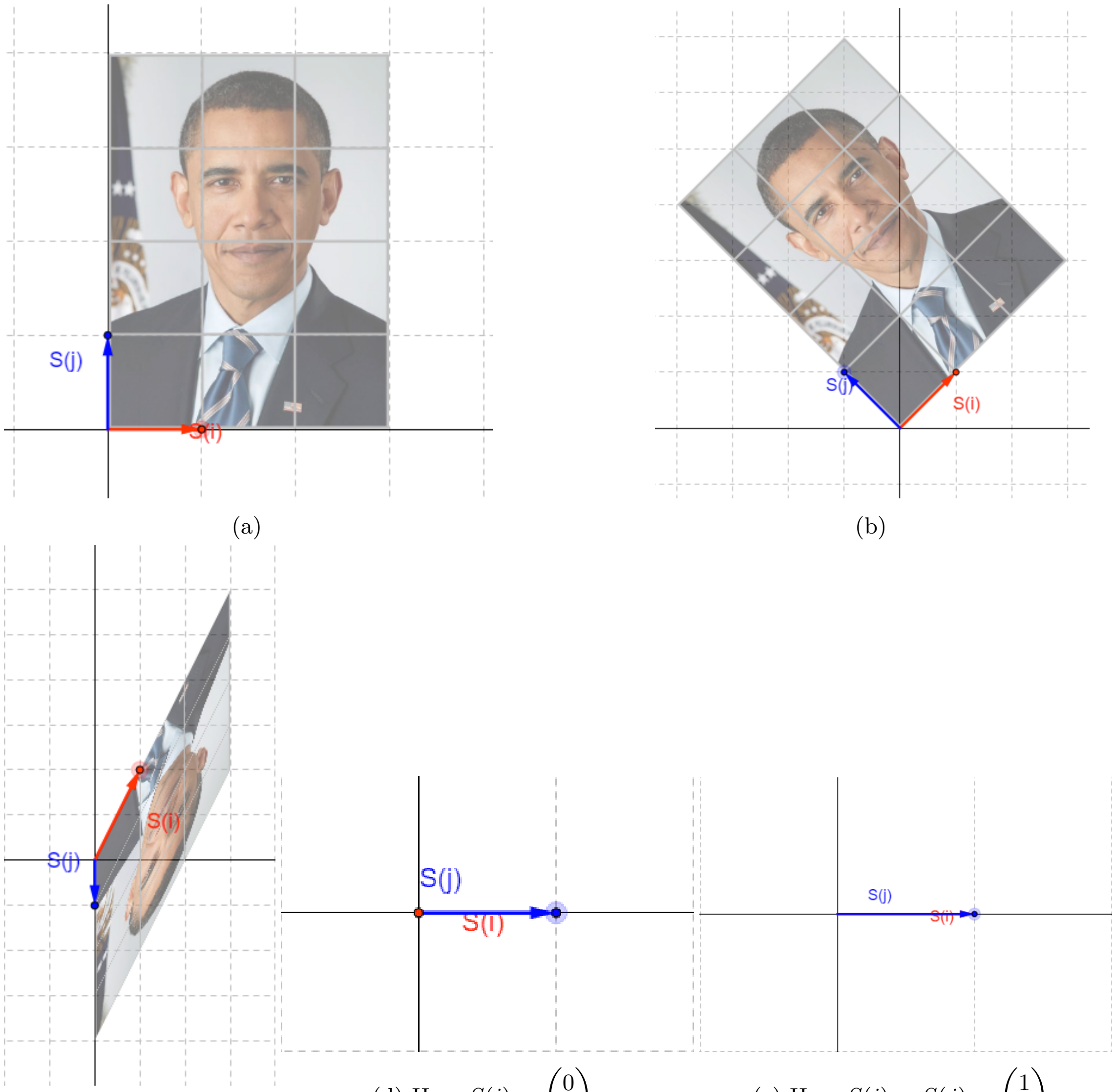


Figure 4



(a)

(b)

(c)

(d) Here $S(i) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(e) Here $S(i) = S(j) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Figure 5: Figures for Exercise 3.

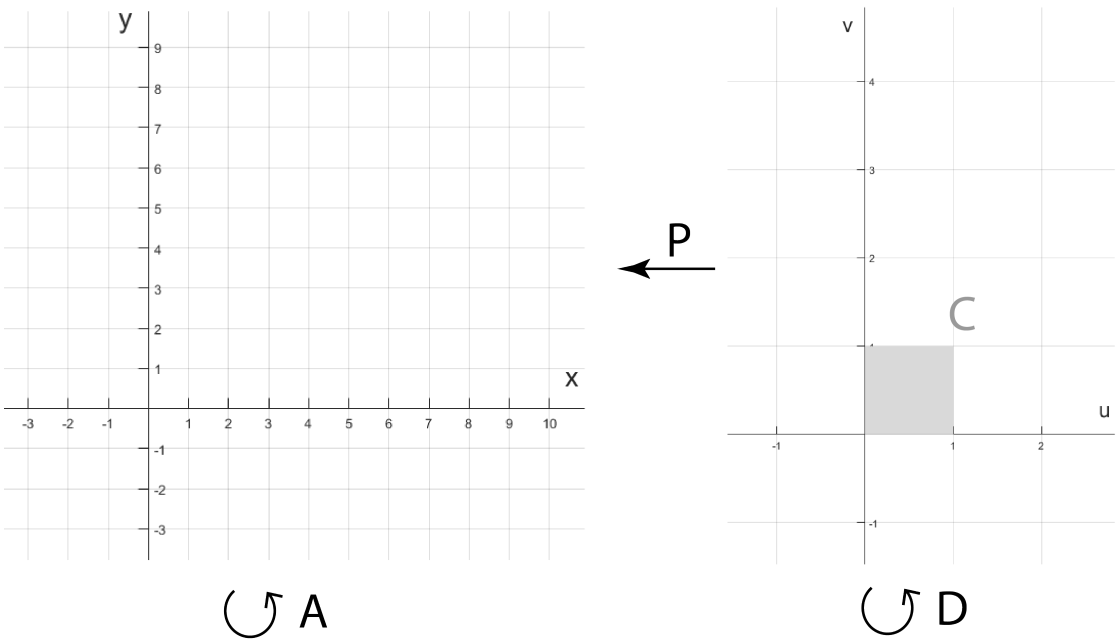


Figure 6: Figure for Exercise 4.

Short answers for exercise 1

1. a) $\partial_x f(x,y) = x^2 - 1$ and $\partial_y f(x,y) = y$, so that $df_{(a,b)}(h,k) = (a^2-1)h + bk$.
   b) $f(a+h, b+k) = f(a,b) + df_{(a,b)}(h,k) + \underset{(h,k)\to(0,0)}{o}(\|(h,k)\|)$.

2. a) $\nabla f(0,1) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. See figure 7.
   b) See figure 7.

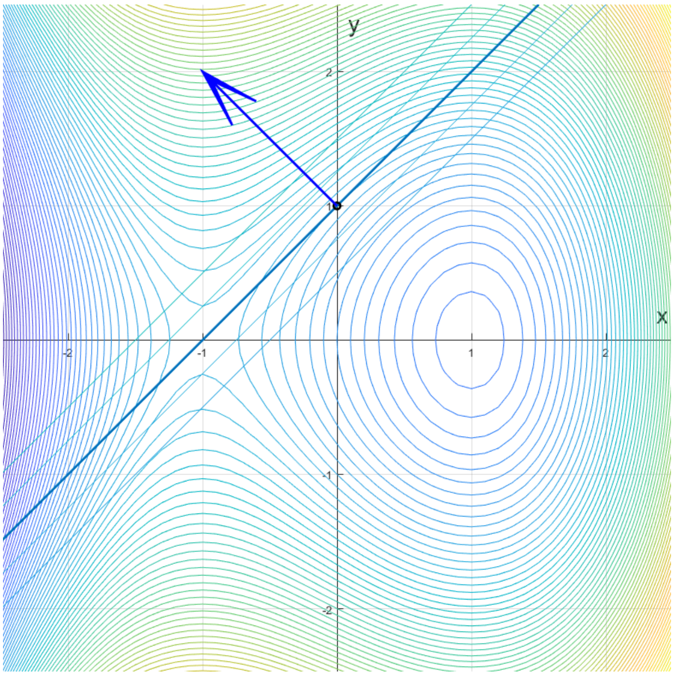


Figure 7: Solution for Exercise 1.

3. a) The matrix of $df_{(0,1)}$ reads: $\begin{pmatrix} -1 & 1 \end{pmatrix}$, the equation of its kernel is: $-h+k=0 \iff h=k$ (or $-x+y=0 \iff x=y$, depending on how you want to denote coordinates),.
   b) $df_{(0,1)}(x-0, y-1) = 0 \iff -x + (y-1) = 0 \iff y = x+1$
   c) $z = df_{(0,1)}(x-0, y-1) \iff z = -x + (y-1) \iff x - y + z = -1$, normal vector: $\begin{pmatrix} 1 & -1 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 1 & -1 \end{pmatrix}$.
   d) The slope is equal to $\|\nabla f(0,1)\| = \sqrt{1+1} = \sqrt{2}$.

Short answers for exercise 2

1. $\mathrm{mat}(DF_{(x,y)}) = \begin{pmatrix} 1 & 1 \\ y & x \end{pmatrix}$, its determinant is equal to $x - y$.

2. a) $\mathcal{F}$ is the line $y = x$ in departure space, $F(x,x) = (2x, x^2)$. See figure 8.
   b) Determinant is positive below the first diagonal, and negative above, see figure 8.
   c) See figure 8. Since $DF_{(1,1)} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, it follows that $\ker(DF_{(1,1)}) = \mathrm{span}(\begin{pmatrix} -1 \\ 1 \end{pmatrix})$ and $\mathrm{im}(DF_{(1,1)}) = \mathrm{span}(\begin{pmatrix} 1 \\ 1 \end{pmatrix})$. As can be seen on the figure, the line passing through $(1,1)$ and parallel to $\ker(DF_{(1,1)})$ is tangent to the level sets of both quantities $x+y$ and xy at the point $(1,1)$ (this is not specific to $(1,1)$, the same is true for every point of the first diagonal).

This is consistent: a direction is in $\ker(DF_{(1,1)})$ if and only if F (that is, the two components $x+y$ and xy of F) remain(s) constant, at first order, under a perturbation in this direction. In other words, $\ker(DF_{(x,y)})$ is not reduced to $0_{\mathbb{R}^2}$ precisely at the points where the level sets of $x+y$ and xy are tangent, and at such points the kernel is directed along this direction of tangency.

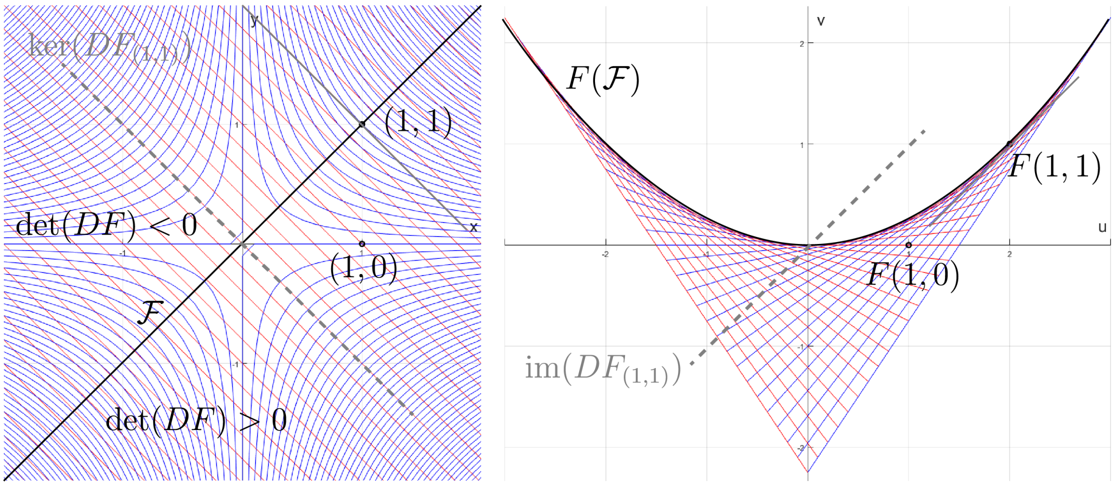


Figure 8: Solution for Exercise 2 (global mapping).

3. a) For every (x, y) in $\mathbb{R}^2$, if $u = x + y$ and $v = xy$, then

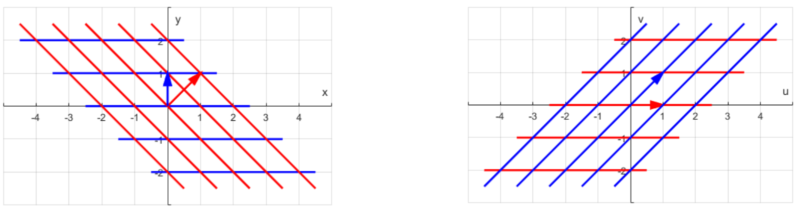
$$u^2 - 4v = (x+y)^2 - 4xy = x^2 + 2xy + y^2 - 4xy = x^2 - 2xy + y^2 = (x-y)^2 \,,$$

and this shows the intended inequality. This inequality is an equality if and only if x equals y.

b) For every (x, x) in $\mathcal{C}$, $F(x, x) = (2x, x^2)$. It follows that $F(\mathcal{C})$ is the parabola defined by the equation $v = u^2/4$. According to the previous question, this is the curve that bounds the image of F (that is, the set $F(\mathbb{R}^2)$), see figure 8.

c) See figure 8. The intended line is tangent to the parabola $F(\mathcal{F})$ at $F(1,1) = (2,1)$. This is consistent, indeed: if this line was not tangent to the parabola, this would show that, for some a small perturbation $(1+h, 1+k)$ of the point $(1,1)$, at first order the image $F(1+h, 1+k)$ would exit the domain below the parabola $v = u^2/4$, a contradiction with the inequality proved in question 3a.

4. a) See figure 9.



(a) (b)

Figure 9

b) Figure to the left of figure 9 can be seen as the limit, when an infinite zoom is applied to figure 2a (departure space) at the point $(x, y) = (1, 0)$. Figure to the right of figure 9 can be seen as the limit, when an infinite zoom is applied to figure 2b (arrival space) at the point $(u, v) = F(1, 0) = (1, 0)$. This is consistent with the appearance of figures 2a, 2b and 9.

Short answers for exercise 3

a) (Identity)

b) $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, conformal, rotation of angle $\pi/4$ composed with homothety of ratio $\sqrt{2}$.

c) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, symmetry with respect to the first diagonal, parallel to ordinate axis.

d) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, nilpotent.

e) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, projection onto the abscissa axis, parallel to the second diagonal.

Short answers for exercise 4

1. $P\mathbf{i} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ thus $AP\mathbf{i} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \frac{1}{2}\ P\mathbf{i}$.

$P\mathbf{j} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ thus $AP\mathbf{j} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 2\ P\mathbf{j}$.

These two vectors are eigenvectors of A, for the eigenvalues $1/2$ and 2, respectively.

2. See figure 10.

3. Since $AP\mathbf{i} = \frac{1}{2}\ P\mathbf{i}$, $P^{-1}AP\mathbf{i} = P^{-1}\frac{1}{2}\ P\mathbf{i} = \frac{1}{2}P^{-1}P\mathbf{i} = \frac{1}{2}\mathbf{i}$. For the same reason, $P^{-1}AP\mathbf{j} = 2\mathbf{j}$.

Thus $D\mathbf{i} = \frac{1}{2}\mathbf{i}$ and $D\mathbf{j} = 2\mathbf{j}$. As a consequence, $D = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$.

4. See figure 10.

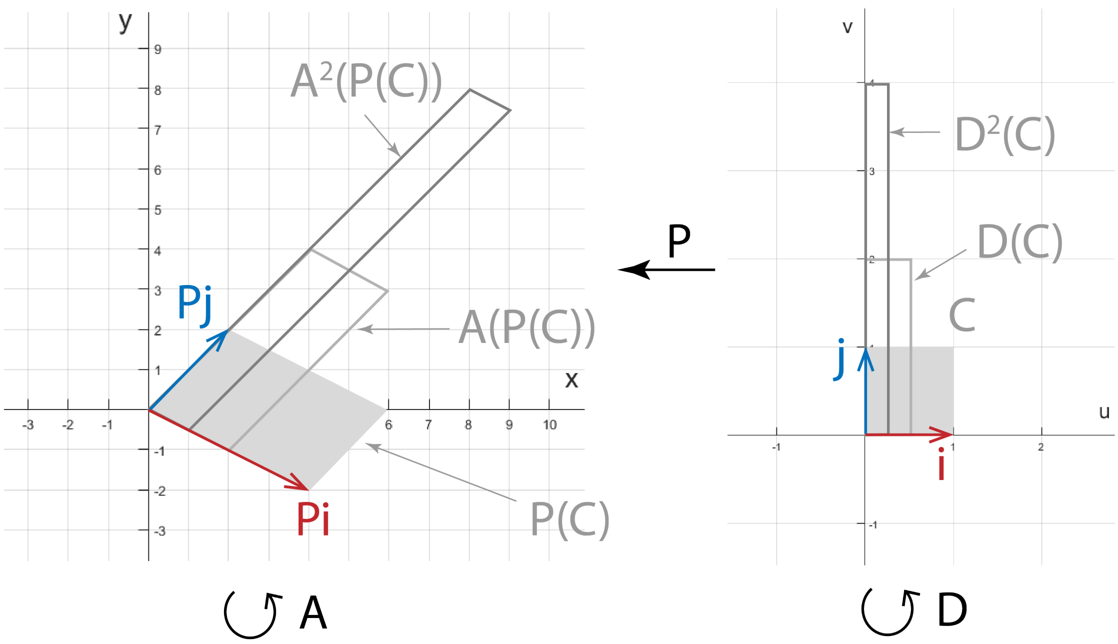


Figure 10: Solution for Exercise 4.

5. a) For vectors of $\text{span}(P\mathbf{i})$.
   b) For vectors of $\text{span}(P\mathbf{j})$.

Short answers for exercise 5

1. The easiest way is to combine rows and columns before doing a Laplace expansion along a column or a row with only one nonzero entry. For instance, $C_3 \leftarrow C_3 + C_1$ and then expansion along the third row gives:

$$\begin{vmatrix} 1 & -1 & 2 \\ 0 & 1 & \alpha \\ -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 3 \\ 0 & 1 & \alpha \\ -1 & 0 & 0 \end{vmatrix} = -\begin{vmatrix} -1 & 3 \\ 1 & \alpha \end{vmatrix} = -(-\alpha - 3) = \alpha + 3\,.$$

2. The matrix A is invertible if and only if its determinant is nonzero, that is if and only if α is not equal to -3. If α is equal to -3, its rank is less than 3 (since the matrix is not invertible) but at least equal to 2 (for instance, the first two columns are linearly independent). It is therefore equal to 2.

Short answers for exercise 6

The values of $\det(A')$ are:

1. $2^n \det(A)$
2. $\det(A)^2$
3. $\det(A)^{-1}$
4. $\det(A)$
5. $\det(A)$
6. $\det(A)$
7. $2\det(A)$
8. $-\det(A)$