Written exam number 3, January 31, 2024

Mathematics, SCAN 2$^{\text{nd}}$ year, 2023–2024

Duration: 2:00.
Five exercises (the statement part is made of **four** pages and the figures part of **five** pages).
No document nor calculation tool allowed.
Vectors are often written (for simplicity and to save space) as rows instead of columns.

Exercise 1. ***Diagonalizability (∼5 points)***

Let us consider the matrix $A = \begin{pmatrix} 2 & -2 & -2 \\ 2 & -2 & -2 \\ -1 & 1 & 2 \end{pmatrix}$, and let us denote by f the corresponding endomorphism of $\mathbb{R}^3$ (the endomorphism of $\mathbb{R}^3$ with matrix A in the canonical basis of $\mathbb{R}^3$).

1. a) Compute the rank of A. Which eigenvalue of A can you derive from this result, and what is its geometric multiplicity?
   b) Compute the characteristic polynomial of A, provide its eigenvalues, and their algebraic multiplicities.
   c) Is A diagonalizable?

2. Let us consider the vectors $u = (1, 1, 0)$, $v = (2, 0, 1)$ and $w = (-1, -1, 1)$.
   a) Show that the family $\mathcal{B} = (u, v, w)$ is a basis of $\mathbb{R}^3$ (you may use a determinant).
   b) Among the vectors u, v, w, which ones are eigenvectors of A? (and for which eigenvalues?)
   c) Let us denote by T the matrix of f in the basis $\mathcal{B}$. Give the expression of T (the calculation of the inverse of the change of basis matrix is *not* required).
   d) Compute T^2; what can you say about the diagonalizability of $f \circ f$?

3. Let M denote an $n \times n$ real matrix. True or false? (carefully justify your answers):
   a) If M is diagonalizable, then so is M^2 (always).
   b) If M^2 is diagonalizable, then so is M (always).

Exercise 2. ***Finding a matrix with a given spectrum (∼3 points)***

1. For each of the six sub-figures in Figure 1, provide (without any justification) a square matrix with real entries, the eigenvalues of which are the points marked in blue (in the complex plane) and such that the two integers next to each blue point (each eigenvalue) represent:

- the algebraic multiplicity (for the first number),
- and the geometric multiplicity (for the second number).

For example, in the third case (c), you need to find a matrix with a single eigenvalue equal to 2, having algebraic multiplicity 3 and geometric multiplicity 2.

2. Consider subfigure (a) of Figure 1 (the first case), and explain how to find *all* 2×2 real matrices with this spectrum. Justify as well as possible.

Exercise 3. ***Two-dimensional linear dynamical systems, real eigenvalues ($\sim$5 points)***

Let us consider the matrix $A = \begin{pmatrix} 7 & 4 \\ -8 & -5 \end{pmatrix}$.

1. What is the sum and the product of the eigenvalues of A, and why?

2. Compute the eigenvalues of A, and, for each of these eigenvalues, provide an eigenvector.

3. Without further justification, provide an invertible 2×2 real matrix P and a diagonal 2×2 matrix D such that $D = P^{-1}AP$.

4. Let us consider the *discrete time* linear dynamical system defined by the matrix A:

$$X_{n+1} = AX_n \,, \quad n \in \mathbb{Z} \,.$$

a) Provide the expressions of two linearly independent eigenmodes for this system, and the expression of any (real) solution (no justification required).

b) Qualify each of the eigenmodes as attractive (stable) or repulsive (unstable) or indifferent (neutral).

c) Without justification, provide the limit and an equivalent of $\|X_n\|$ as n goes to $+\infty$ (depending on the initial condition X_0).

5. Let us consider the *continuous time* linear dynamical system defined by the matrix A:

$$\dot{X}(t) = AX(t) \,, \quad t \in \mathbb{R} \,.$$

a) Provide the expressions of two linearly independent eigenmodes for this system, and the expression of any (real) solution (no justification required).

b) Qualify each of the eigenmodes as stable (attractive) or unstable (repulsive) or neutral (indifferent).

c) On Figure 2, draw: to the right, the phase portrait of the linear dynamical system $\dot{U} = DU$ ($U = (u, v)$) and to the left, the phase portrait of the linear dynamical system $\dot{X} = AX$ ($X = (x, y)$). No justification is required, but the quality and accuracy of the drawings will be taken into account.

d) Without justification, provide the limit and an equivalent of $\|X(t)\|$ as t goes to $+\infty$ (depending on the initial condition $X(t = 0)$).

Exercise 4. ***Two-dimensional linear dynamical systems, complex eigenvalues (~4 points)***

1. Let us consider the matrix $A = \begin{pmatrix} 1 & 1 \\ -5 & -1 \end{pmatrix}$.

a) Compute the eigenvalues λ and $\bar{\lambda}$ of this matrix.

b) Provide an eigenvector w, for the eigenvalue λ, *of the form* $(1, b)$ (provide the value of b).

c) Without further justification, provide the conformal matrix C defined by $C = Q^{-1}AQ$, where Q is the change of basis matrix, from the canonical basis of $\mathbb{R}^2$ to the basis $\big(\mathrm{Re}(w), \mathrm{Re}(iw)\big)$.

2. Let us consider the *discrete time* linear dynamical systems defined by the matrices A and C:

$$X_{n+1} = AX_n\,, \quad n \in \mathbb{Z}\,, \quad \text{and} \quad U_{n+1} = CU_n\,, \quad n \in \mathbb{Z}\,,$$

and, more specifically, let us consider the sequences $(X_n)_{n\in\mathbb{Z}}$ and $(U_n)_{n\in\mathbb{Z}}$ defined by the initial conditions: $U_0 = (0, 1)$ (warning: not $(1, 0)$!) and $X_0 = QU_0$.

a) Recall why $X_n = QU_n$, for every n in $\mathbb{Z}$.

b) On Figure 3, draw:

- to the right, the points U_n for n in $\{-1, \dots, 3\}$,
- and to the left, the points X_n, for n in $\{-1, \dots, 3\}$.

c) Without further justification, provide the values of the limits

$$\lim_{n\to+\infty} \|U_n\| \quad \text{and} \quad \lim_{n\to+\infty} \|X_n\| \quad \text{and} \quad \lim_{n\to-\infty} \|U_n\| \quad \text{and} \quad \lim_{n\to-\infty} \|X_n\|\ .$$

3. Let us keep the previous notation, and let us consider the *continuous time* linear dynamical systems defined by the matrices A and C:

$$\dot{X}(t) = AX(t)\,, \quad t \in \mathbb{R}\,, \quad \text{and} \quad \dot{U}(t) = AU(t)\,, \quad t \in \mathbb{R}\,.$$

a) To the right of Figure 4, draw the phase portrait of the system $\dot{U} = CU$ (in particular, draw the trajectory passing through the point $(1, 0)$).

b) To the left of Figure 4, draw the image by Q of the square centred at the origin drawn in the (u, v) plane, and then draw (as well as possible) the phase portrait of the system $\dot{X} = AX$.

Exercise 5. ***Nonlinear differential system (∼3 points off bonus)***
Let us consider the function

$$F : \mathbb{R}^2 \to \mathbb{R}^2\,, \quad X = \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ -2x + 2x^3 - 3y \end{pmatrix}\,.$$

1. For which values of (x, y) do we have $F(x, y) = (0, 0)$?
2. Compute the differential $DF_{(x,y)}$, and then $DF_{(0,0)}$, and $DF_{(1,0)}$ (which is equal to $DF_{(-1,0)}$).
3. The eigenvalues of $DF_{(0,0)}$ are -2 and -1, and the eigenvalues of $DF_{(1,0)}$ are -4 and 1. How can you fast-check that these values are valid? Provide, for each of these two matrices and for each eigenvalue, an eigenvector.
4. Figure 5 represents the phase portrait of the (nonlinear) differential system

$$\dot{X}(t) = F\big(X(t)\big) \tag{1}$$

(the three red dots are the points $(-1, 0)$, $(0, 0)$, and $(1, 0)$). Below this figure, provide a rough drawing of the two phase portraits of the linear differential systems

$$\dot{X}(t) = DF_{(0,0)}X(t) \quad \text{and} \quad \dot{X}(t) = DF_{(\pm1,0)}X(t)\,,$$

and explain how these two phase portraits are related to the phase portrait of system (1).
5. [Bonus] Let us consider the functions

$$V : \mathbb{R} \to \mathbb{R}\,, \quad x \mapsto x^2 - \frac{x^4}{2}\,,$$
$$\text{and} \quad E : \mathbb{R}^2 \to \mathbb{R}\,, \quad X = \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \frac{y^2}{2} + V(x)\,,$$

and the second order differential equation

$$\ddot{x} = -3\,\dot{x} - 2x + 2x^3 = -3\,\dot{x} - V'(x)\,, \tag{2}$$

which is equivalent to the differential system (1).

a) Assuming that $t \mapsto x(t)$ is a solution of this differential equation (2), compute the quantity

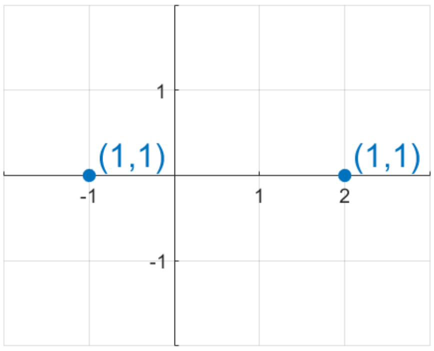
$$\frac{d}{dt}E\big(x(t), \dot{x}(t)\big)$$

(hint: multiply the differential equation by $\dot{x}$). What does $E\big(x(t), \dot{x}(t)\big)$ represent? What do the terms $-3\dot{x}$ and $-V'(x)$ represent in the differential equation?

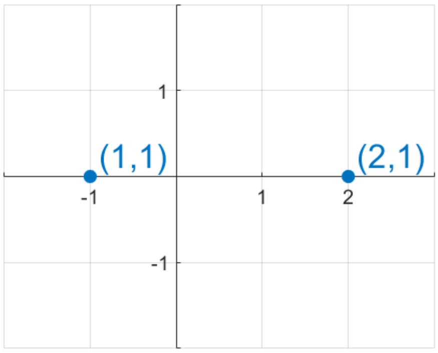
b) Draw the graph of $x \mapsto V(x)$, and provide a mechanical interpretation of the solutions satisfying $X(t) \to (1, 0)$ as $t \to +\infty$, and of nearby solutions.

First name: Last name:

If you need another print of this “figures” part (to change your answer after some mistake), feel free to ask!



(a)



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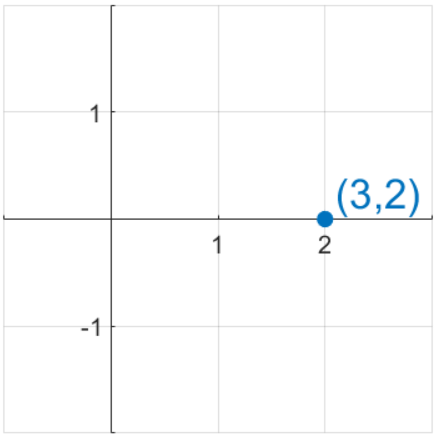
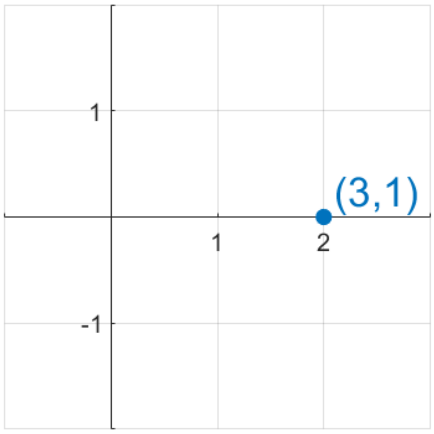


(c)



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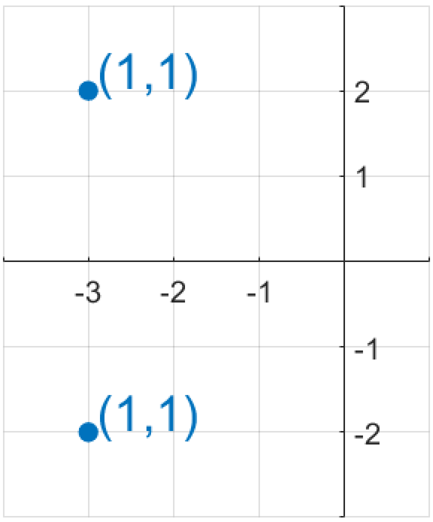

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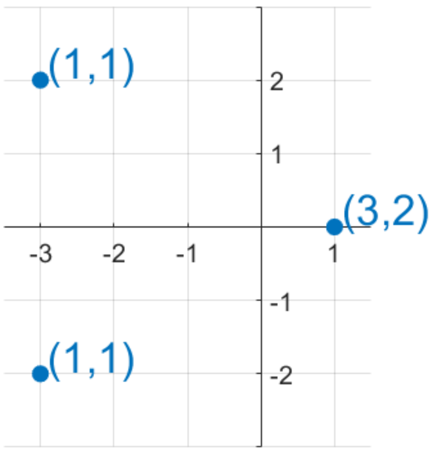

(f)

Figure 1: Figure for Exercise 2.

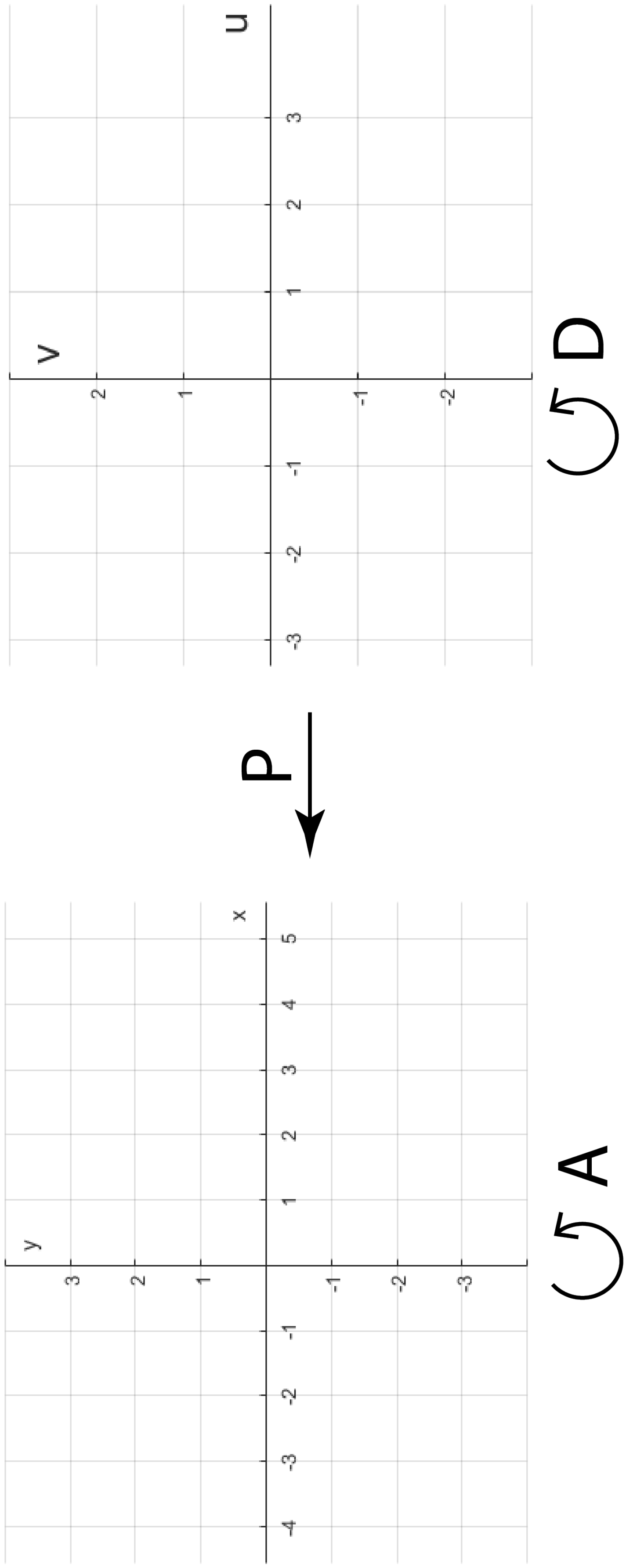

Figure 2: Figure for Exercise 3 (real eigenvalues, continuous time).

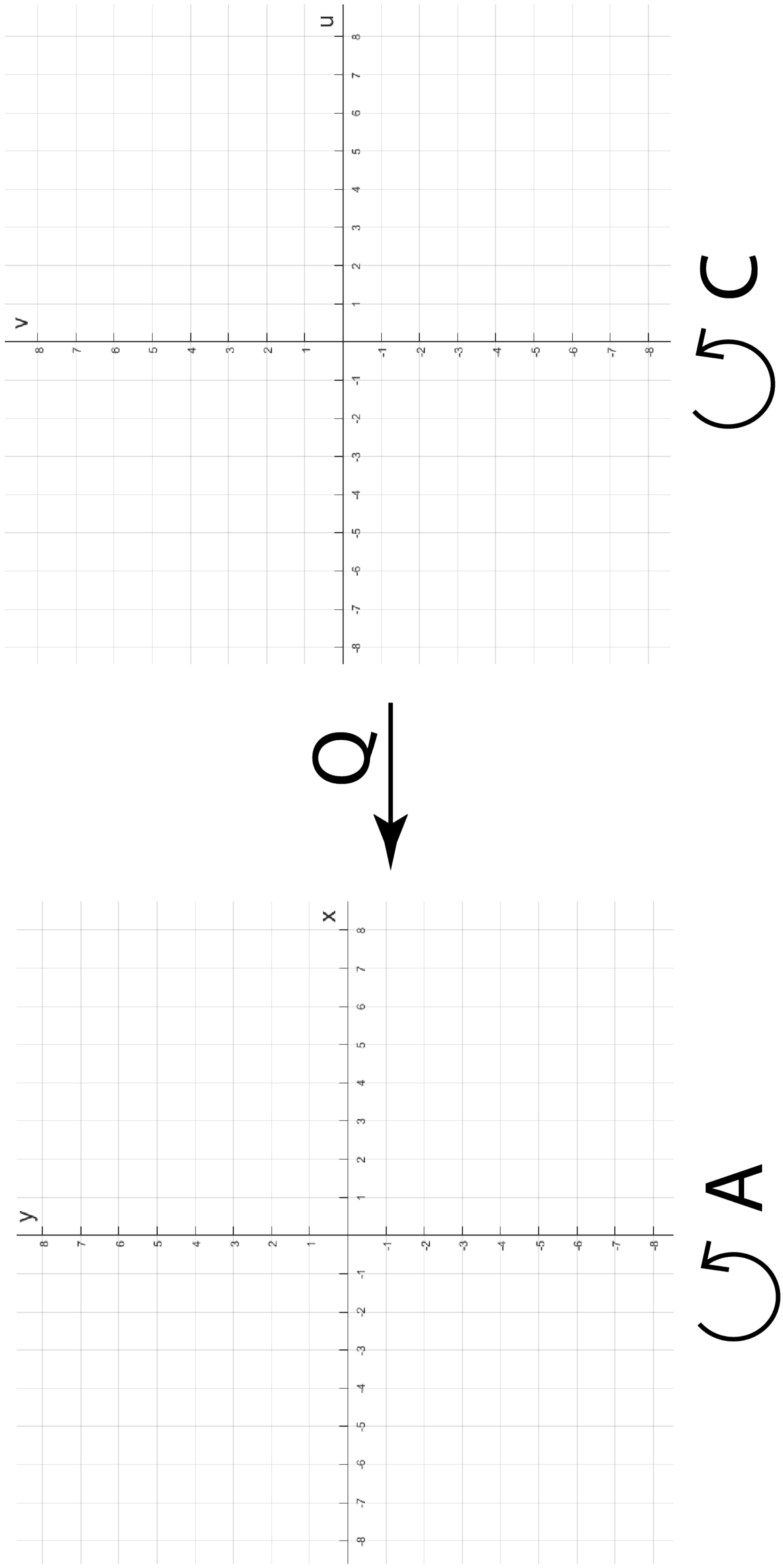

Figure 3: First figure for Exercise 4 (complex eigenvalues, discrete time).

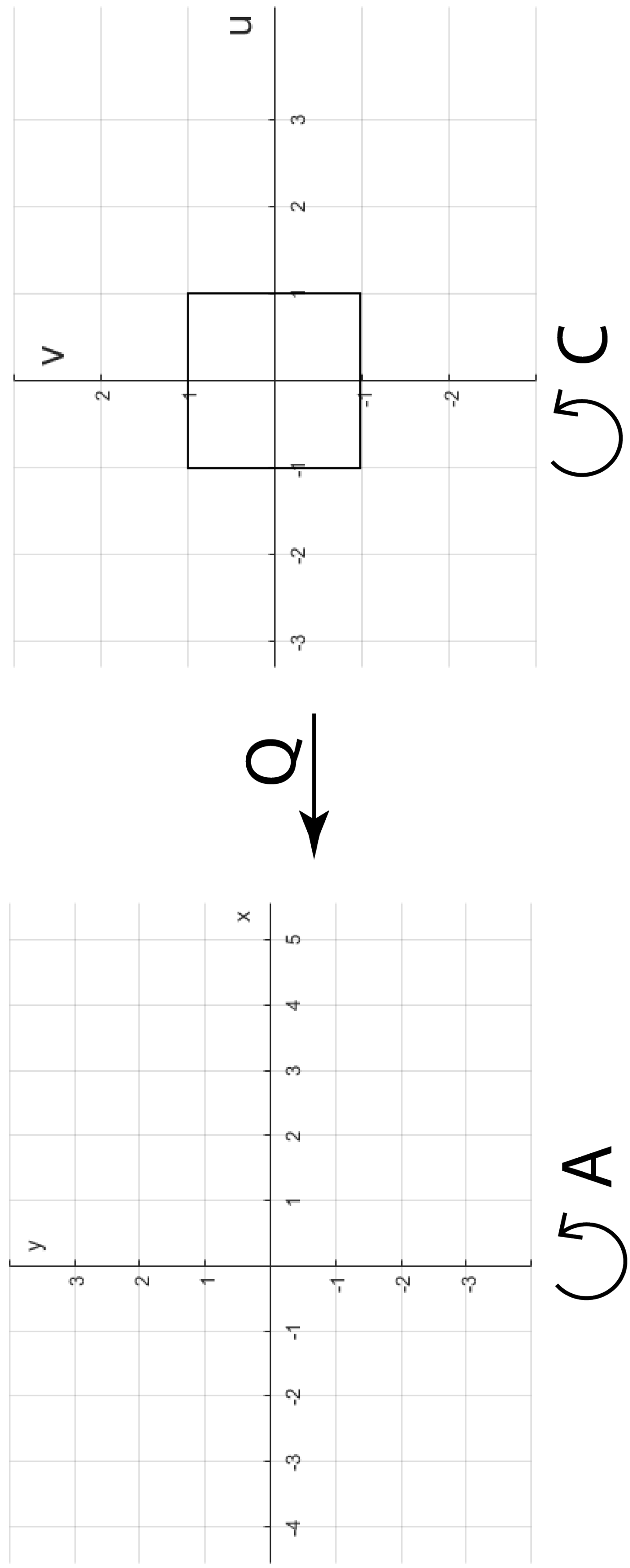

Figure 4: Second figure for Exercise 4 (complex eigenvalues, continuous time).

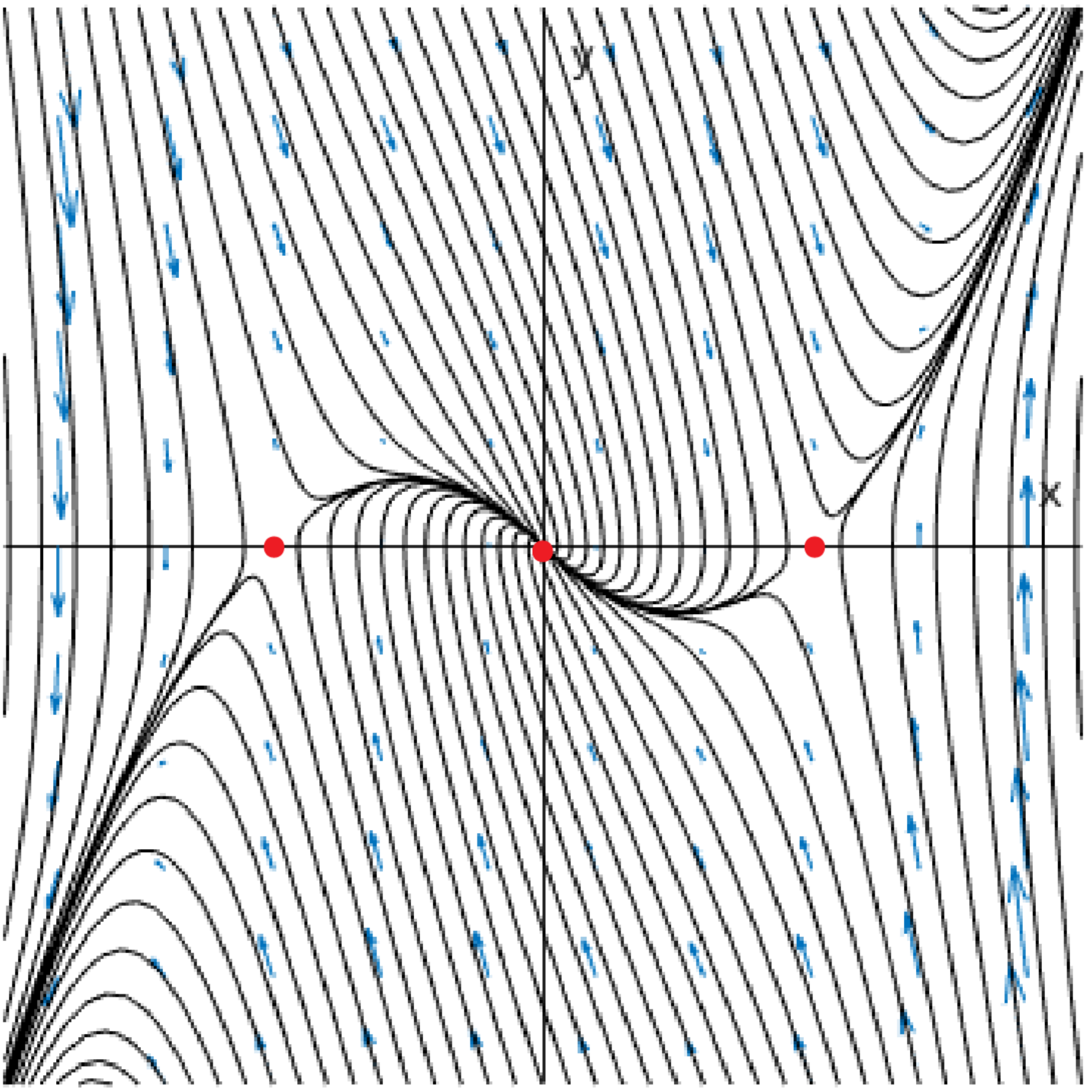

Figure 5: Figure for Exercise 5.

Short answers for exercise 1

1. a) $\mathrm{rank}(A) = 2$ (for instance, the two first rows are equal, and the two last rows are linearly independent). Thus $\dim\big(\ker(A)\big) = 1$, so that 0 is an eigenvalue for A, with geometric multiplicity 1.

b) $P_A(\lambda) = -\lambda^2(\lambda - 2)$ (several ways to carry out the computation). Thus the eigenvalues are 0 and 2, with respective algebraic multiplicities 2 and 1.

c) Since the sum of the geometric multiplicities is $1 + 1 = 2$ (and not 3), the matrix A is not diagonalizable.

2. a) The determinant of (u, v, w) is equal to -2 (after computation), thus the family is a basis of $\mathbb{R}^3$ (other method: put the matrix of the family in echelon form).

b) By direct computation $Au = (0, 0, 0)$ and $Av = (2, 2, 0) = 2u$ and $Aw = (-2, -2, 2) = 2w$. Thus u and w are eigenvectors for the eigenvalues 0 and 2, respectively (and v is not an eigenvector).

c) According to the previous computations, $T = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

d) By direct computation, $T^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$, which is diagonal, and which is the matrix of f in the basis $\mathcal{B}$. Thus $f \circ f$ is diagonalizable.

3. a) True: if M is diagonalizable then there exists an invertible matrix P and a diagonal matrix D such that $M = PDP^{-1}$, so that $M^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$, and since D is diagonal so is D^2, so that M^2 is diagonalizable.

b) False: in the example above A^2 is diagonalizable, whereas A is not.

Short answers for exercise 2

1. a) $\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$

b) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

c) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

d) $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

e) $\begin{pmatrix} -3 & -2 \\ 2 & -3 \end{pmatrix}$

f) $\begin{pmatrix} -3 & -2 & 0 & 0 & 0 \\ 2 & -3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

2. If a 2×2 real matrix A has this spectrum, then it is diagonalizable (on $\mathbb{R}$), thus if we denote by D the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$, then there exists an invertible 2×2 real matrix P such that $A = PDP^{-1}$. On the other hand, every matrix of the form PDP^{-1} has the same spectrum as D, thus the spectrum of case (a). In short, the 2×2 real matrices having this spectrum are exactly the matrices of the form PDP^{-1}, where P is a 2×2 invertible real matrix.

Short answers for exercise 3

1. The sum of the eigenvalues is equal to the trace, which is $7 - 5 = 2$, and the product of the eigenvalues is the determinant, which is equal to $-35 + 32 = -3$.

2. $P_A(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 - 2\lambda - 3$, discriminant equals 16, the eigenvalues are therefore

$$\frac{2 \pm 4}{2} = 1 \pm 2\,, \quad \text{that is} \quad \lambda_- = -1 \quad \text{and} \quad \lambda_+ = 3\,.$$

$A - (-1)I_2 = \begin{pmatrix} 8 & 4 \\ -8 & -4 \end{pmatrix}$ and $A - 3I_2 = \begin{pmatrix} 4 & 4 \\ -8 & -8 \end{pmatrix}$. Possible (respective) eigenvectors are therefore: $w_- = (-1, 2)$ and $w_+ = (-1, 1)$. Thus the equality $D = P^{-1}AP$ holds for $P = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$.

3. a) According to the course, the expression of the two eigenmodes is (up to multiplication by a scalar factor):

$$n \mapsto \lambda_-^n w_- \quad \text{and} \quad n \mapsto \lambda_+^n w_+\,,$$

and any solution is a linear combination of these two eigenmodes:

$$n \mapsto C_- \lambda_-^n w_- + C_+ \lambda_+^n w_+\,, \quad (C_-, C_+) \in \mathbb{R}^2\,.$$

b) The eigenmode $n \mapsto \lambda_-^2 w_-$ is indifferent and the eigenmode $n \mapsto \lambda_+^n w_+$ is repulsive.

c) If X_0 is not in $\operatorname{span}(w_-)$ (in other words if the coefficient C_+ associated with this solution is nonzero), then $\|X_n\| \to +\infty$ as $n \to +\infty$, and $\|X_n\| \sim |C_+|\, \lambda_+^n \, \|w_+\|$ as $n \to +\infty$. Conversely, if X_0 is in $\operatorname{span}(w_-)$, then $X_n = \pm C_- w_-$, so that $\|X_n\|$ is constant with n and equal to $|C_-|\, \|w_-\|$ (this is quantity is therefore its limit and equivalent as $n \to +\infty$).

4. a) According to the course, the expression of the two eigenmodes is (up to multiplication by a scalar factor):

$$t \mapsto e^{t\lambda_-} w_- \quad \text{and} \quad t \mapsto e^{t\lambda_+} w_+\,,$$

and any solution is a linear combination of these two eigenmodes:

$$n \mapsto C_- e^{t\lambda_-} w_- + C_+ e^{t\lambda_+} w_+\,, \quad (C_-, C_+) \in \mathbb{R}^2\,.$$

b) The eigenmode $t \mapsto e^{t\lambda_-} w_-$ is stable and the eigenmode $e^{t\lambda_+} w_+$ is unstable.

c) See Figure 6. Reverting the order between the two eigenvectors results in a diagonal matrix D equal to $\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$, and a picture to the right which is flipped with respect to the first diagonal, while picture to the left is unchanged.

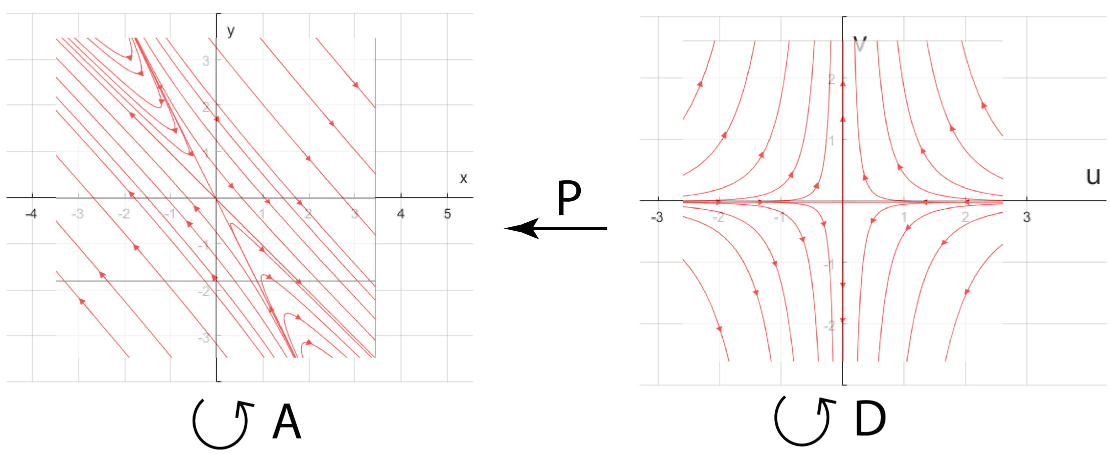

Figure 6: Solution for question 4c of Exercise 3.

d) If $X(0)$ is not in $\operatorname{span}(w_-)$ (in other words if the coefficient C_+ associated with this solution is nonzero), then $\|X(t)\| \to +\infty$ as $t \to +\infty$, and $\|X(t)\| \sim |C_+|\, e^{t\lambda_+} \, \|w_+\|$ as $t \to +\infty$. Conversely, if X_0 is in $\operatorname{span}(w_-)$ (in other words if the coefficient C_+ is zero), then $\|X(t)\| \to 0$ and $\|X(t)\| \sim |C_-|\, e^{t\lambda_-} \, \|w_-\|$ as $t \to +\infty$.

Short answers for exercise 4

1. a) $T = 1 - 1 = 0$ and $D = -1 + 5 = 4$, so that $P_A(X) = X^2 + 4$, the two eigenvalues are therefore $\lambda = 2i$ and $\overline{\lambda} = -2i$.

   b) $A - 2iI_2 = \begin{pmatrix} 1-2i & 1 \\ -5 & -1-2i \end{pmatrix}$, so that $w = (1, -1+2i)$ is an eigenvector for the eigenvalue $2i$. According to the course, if $Q = \begin{pmatrix} 1 & 0 \\ -1 & -2 \end{pmatrix}$, then $C = Q^{-1}AQ = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$, which is the conformal matrix corresponding to the multiplication by $2i$ in the complex plane.

2. a) $X_n = A^n X_0 = (QCQ^{-1})^n X_0 = QC^nQ^{-1}X_0 = QC^nU_0 = QU_n$.

   b) See Figure 7.

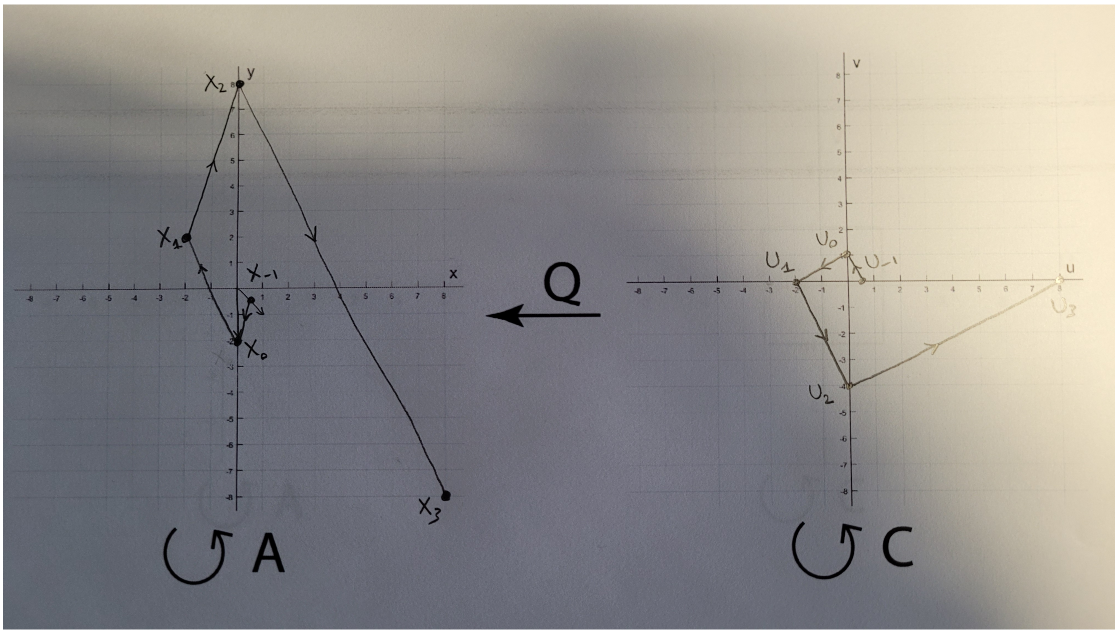

Figure 7: Solution for question 2a of Exercise 4.

   c) The two fist limits are equal to $+\infty$, and the two last limits are equal to 0 (except if $X_0 = (0, 0)$, in this case $X_n = (0, 0)$ for all n in $\mathbb{Z}$ so that the two limits are equal to 0).

3. a) See Figure 8.

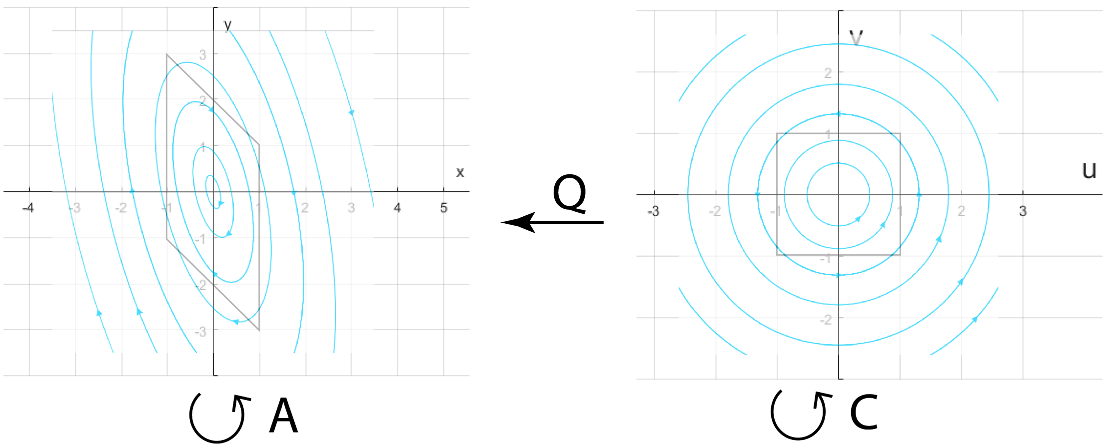

Figure 8: Solution for question 3 of Exercise 4.

   b) See Figure 8.

Short answers for exercise 5

1. $F(x,y)=(0,0) \iff y=0$ and $-x+x^3=0$, and this last condition is equivalent to $x=0$ or $x=\pm 1$. The three solutions are therefore $(-1,0)$, $(0,0)$, and $(0,1)$.

2. $DF_{(x,y)} = \begin{pmatrix} 0 & 1 \\ -2+6x^2 & -3 \end{pmatrix}$, so that $DF_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$ and $DF_{(1,0)} = DF_{(-1,0)} = \begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix}$.

3. In each case, the sum of the given values equals the trace, and their product equals the determinant, which confirms that these given values are indeed the eigenvalues of the matrix. In each case, an eigenvector is $(1,\lambda)$, where λ is the eigenvalue (this is always the case for a companion matrix, and $DF_{(x,y)}$ is a companion matrix).

4. See Figure 9. As shown on the figure, locally around every equilibrium point (the red dots), the phase portrait of the nonlinear system is close (closer and closer if considered more and more locally) to the phase portrait of the linear system associated with the differential at the equilibrium (the linearization of the nonlinear system at this point).

5. a) By multiplying the differential equation (2) by $\dot{x}$, we get immediately: $\frac{d}{dt}E\big(x(t),\dot{x}(t)\big) = -3\ \dot{x}(t)^2$. The quantity $E\big(x(t),\dot{x}(t)\big)$ represents the (mechanical) energy of the solution, it decreases with time due to the damping, the damping force is the term $-3\dot{x}$, and the term $-V'(x)$ is minus the gradient of the potential, it represents a (conservative) force deriving from the potential V (conservative means that this force does not change the mechanical energy of a solution, whereas the damping force does).

b) See Figure 10. At -1 and $+1$ the potential reaches a maximum. Solutions that converge exactly to $(1,0)$ have exactly the right impulsion to reach this maximum of the potential, nearby solution have (on one side) a slightly smaller impulsion (not enough impulsion to reach this maximum of the potential) or have (on the other side) a slightly greater impulsion, enough to pass (cross) this maximum of the potential.

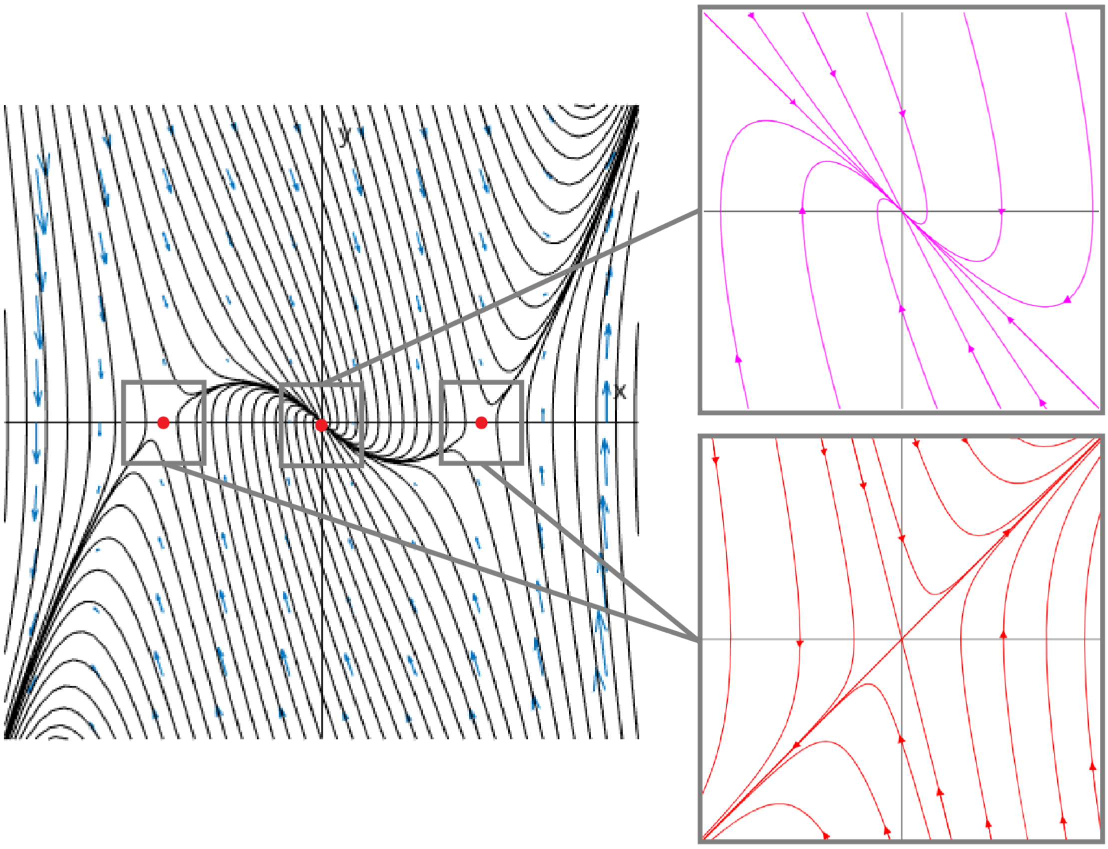

Figure 9: Solution for question 4 of Exercise 5.

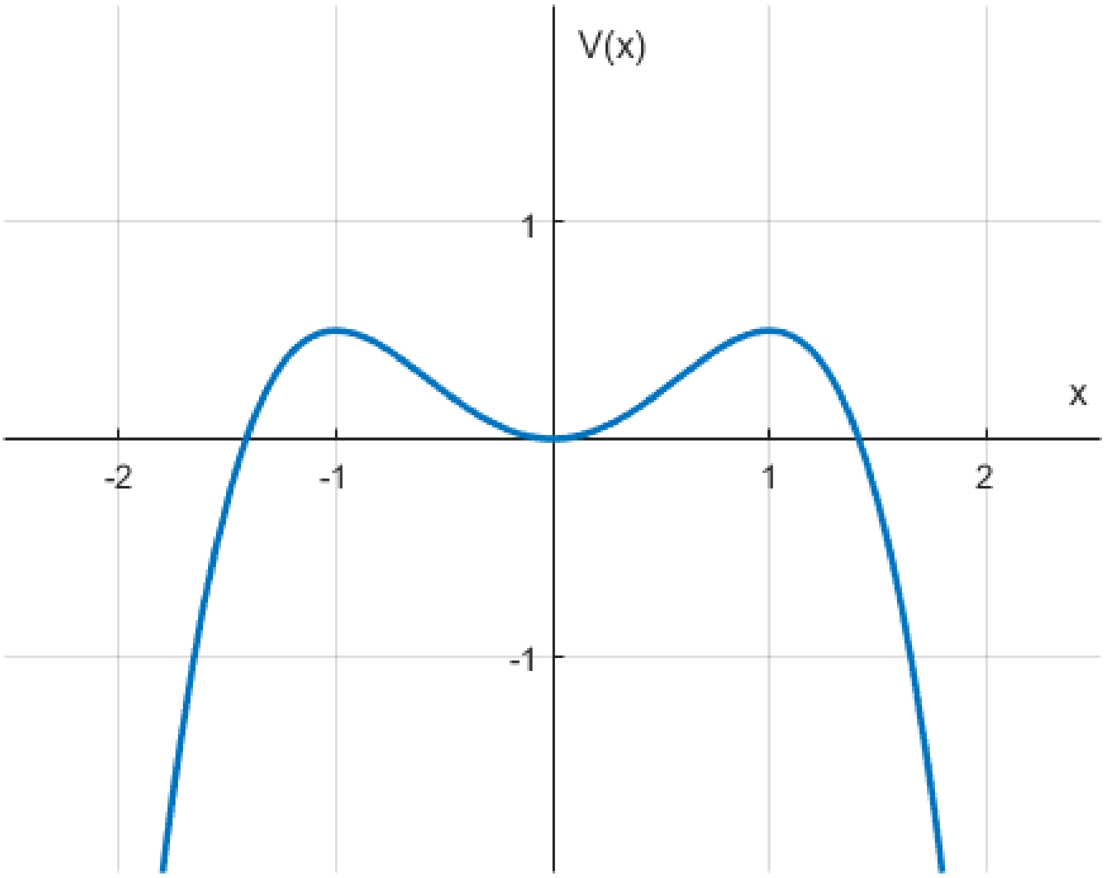

Figure 10: Solution for question 5 of Exercise 5.