Written exam number 4, April 29, 2024

Mathematics, SCAN 2ND year, 2023–2024

Duration: 2:00.
No document nor calculation tool allowed.

Exercise 1. ***Power series (∼5.5 points)***

1. Let us consider the differential equation:

$$xy''(x) - y(x) = 0\,,$$

and a power series $\displaystyle\sum_{n=0}^{+\infty} a_n x^n$, with a positive radius of convergence R, and let us assume that the function $y : (-R, R) \to \mathbb{R}$, $x \mapsto y(x)$, defined as the sum of this power series, is a solution of this differential equation.

a) Find a recurrence relation satisfied by the coefficients a_n, and provide the value of a_0.
b) For every integer n greater than or equal to 1, express a_n as a function of n and a_1.
c) What is the value of R?

2. Let us consider the function f defined as the sum of the power series $\displaystyle\sum_{n=1}^{+\infty} \frac{x^n}{n!(n-1)!}$.

a) What are the values of $f(0)$, $f'(0)$, and $f''(0)$? derive from these values a rough drawing of the graph of f over a small neighbourhood of $x = 0$.
b) Recall the upper bound on the absolute value of the remainder for a numerical series satisfying the alternating series test, and explain why the quantity $-1 + \frac{1}{2} - \frac{1}{12} = -\frac{7}{12}$ approximates $f(-1)$ up to an error not exceeding 10^{-2}.
c) Prove that f is increasing (meaning: "strictly" increasing) on $[0, +\infty)$.
d) Prove that, for all x in $[0, +\infty)$, $f(x)$ is greater than or equal to x. What is the value of $\lim_{x\to+\infty} f(x)$?

Exercise 2. ***Sequences (∼6 points)***

Let us consider the two (odd) functions f and g, from $\mathbb{R}$ to $\mathbb{R}$, defined as:

$f(x) = -ax + x^3$ and $g(x) = -bx + x^3$, for some real quantities a and b satisfying $0 < a < 1$ and $1 < b$.

The graphs of these two functions, together with the graphs of $x \mapsto x$ and $x \mapsto -x$, are shown on Figures 1a and 1b. The goal of the exercise is to study sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ defined by the recurrence relations

$$u_{n+1} = f(u_n) \quad \text{and} \quad v_{n+1} = g(v_n)\,,$$

depending on the values of the initial conditions u_0 and v_0. The following facts are admitted (α, β, and γ are positive quantities, see Figures 1a and 1b):

$$\begin{aligned} & f(x) = x \iff x \in \{-\alpha, 0, \alpha\} \quad \text{and} \quad f(x) = -x \iff x = 0\,, \\ \text{and}\quad & g(x) = x \iff x \in \{-\beta, 0, \beta\} \quad \text{and} \quad g(x) = -x \iff x \in \{-\gamma, 0, \gamma\}\,, \\ \text{and}\quad & 1 < f'(\alpha) \quad \text{and} \quad 1 < g'(\beta) \quad \text{and} \quad -1 < g'(\gamma) < 0\,. \end{aligned}$$

1. Which qualifiers (attractive / repulsive) apply to the fixed points of f and g and why?

2. Describe the behaviour of the sequence $(v_n)_{n\in\mathbb{N}}$ for $v_0 = \gamma$. Which qualifier (attractive / repulsive) would *you* apply to this sequence and why?

3. In this question, only the drawings and the answers are required, without any further justification. *Warning: the graphs of $x \mapsto -x$ are here to help, but should not be directly used for the drawings of scales!*

a) Describe the various possible asymptotic behaviours (that is, as n goes to $+\infty$) of the sequence $(u_n)_{n\in\mathbb{N}}$, depending on the initial condition u_0, and illustrate these behaviours (drawing of "scales") on Figure 1a.
b) Same question for the second function/figure: describe the various possible asymptotic behaviours of the sequence $(v_n)_{n\in\mathbb{N}}$, depending on the initial condition v_0, and illustrate these behaviours (drawing of "scales") on Figure 1b.

4. Using the fact that $f(x) - x$ is positive for all x in $(\alpha, +\infty)$ (no need to prove this fact), provide a rigorous proof of the asymptotic behaviour of the sequence $(u_n)_{n\in\mathbb{N}}$ when u_0 belongs to this interval.

5. Let us assume that u_0 is in $(-\alpha, \alpha) \setminus \{0\}$, and let us consider the sequence $(w_n)_{n\in\mathbb{N}}$ defined as $w_n = |u_n|$. Using the fact that the function $x \mapsto |f(x)|$ is continuous and that, for every n in $\mathbb{N}$, $w_{n+1} = |f(w_n)| < w_n$ (no need to prove these facts), provide a rigorous proof of the asymptotic behaviour of the sequence $(w_n)_{n\in\mathbb{N}}$, and consequently of the sequence $(u_n)_{n\in\mathbb{N}}$.

6. Let us assume that u_0 belongs to $(-\alpha, \alpha) \setminus \{0\}$.

a) Using the Mean Value Theorem, prove that there exists a sequence $(z_n)_{n\in\mathbb{N}}$, going to 0 as n goes to $+\infty$, such that, for every positive integer n,

$$|u_n| = |u_0| \times \prod_{j=0}^{n-1} |f'(z_j)| = |u_0| \times |f'(z_0)| \times \cdots \times |f'(z_{n-1})| \ .$$

b) [Bonus] Provide the value of the limit: $\displaystyle\lim_{n\to+\infty} \frac{1}{n} \ln |u_n|$, and a rough justification of this value.

Exercise 3. ***Inner products, 1 (∼5 points)***
Let us denote by E the space $\mathbb{R}_3[X]$ and let us consider the function $\varphi : E \times E \to \mathbb{R}$, defined as: $\varphi(P, Q) = \displaystyle\int_{-1}^{1} P(t)Q(t)\, dt$.

1. Recall the definition (the required properties and their precise meaning) of an inner product, and prove that φ is positive definite, and is therefore an inner product on E (the proof of the other properties is obvious an can be skipped). Let us denote by $\|\cdot\|$ the corresponding Euclidean norm.

2. a) Prove the following inequality: for every P in $\mathbb{R}_3[X]$,

$$\frac{1}{2}\left(\int_{-1}^{1} P(t)\, dt\right)^2 \le \int_{-1}^{1} P(t)^2\, dt \, .$$

b) For which polynomials of $\mathbb{R}_3[X]$ is this inequality actually an equality?

3. Let us consider the set $F = \left\{ P \in \mathbb{R}_3[X] : \displaystyle\int_{-1}^{1} (t^3 - t)P'(t)\, dt = 0 \right\}$.

a) Prove that, for every P in $\mathbb{R}^3[X]$,

$$P \in F \iff P \perp 3X^2 - 1 \, .$$

b) The following facts are admitted: F is a vector subspace of E, and $F^\perp = \operatorname{span}(3X^2 - 1)$. Let us consider the polynomial $Q = X + 2X^2$. Compute the quantity

$$\operatorname{dist}(Q, F) = \inf_{P\in F} \|Q - P\| \ .$$

Exercise 4. ***Inner products, 2 (∼3.5 points)***
Let us denote by E the space $\mathcal{C}^0\big([0, \pi], \mathbb{R}\big)$ and let us consider the function $\varphi : E \times E \to \mathbb{R}$, defined as: $\varphi(f, g) = \displaystyle\int_0^{\pi} f(t)g(t)\sin(t)\, dt$. The following three facts are admitted:

1. φ is an inner product on E,
2. $\displaystyle\int_0^{\pi} t\sin(t)\, dt = \pi$,
3. $\displaystyle\int_0^{\pi} t^2 \sin(t)\, dt = \pi^2 - 4$.

Let us consider the functions u_1, u_2, and u_3 of E, defined as:

$$u_1(t) = 1 \quad \text{and} \quad u_2(t) = t \quad \text{and} \quad u_3(t) = \cos(t)\, ,$$

and let $F = \operatorname{span}(u_1, u_2)$ (that is, F is the subspace of affine functions in E).

1. Find an *orthogonal* basis of F (no need to normalize its vectors).

2. Among the functions of F, find the one which is the closest to u_3 (that is, closest to the function cos) for the Euclidean norm associated with φ.

Written exam number 4, April 29, 2024 (Figures)

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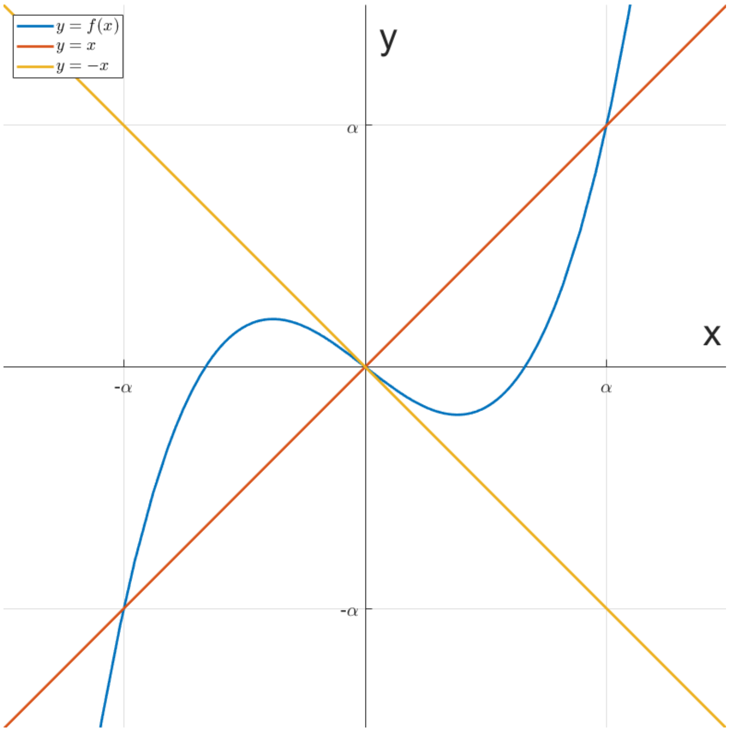
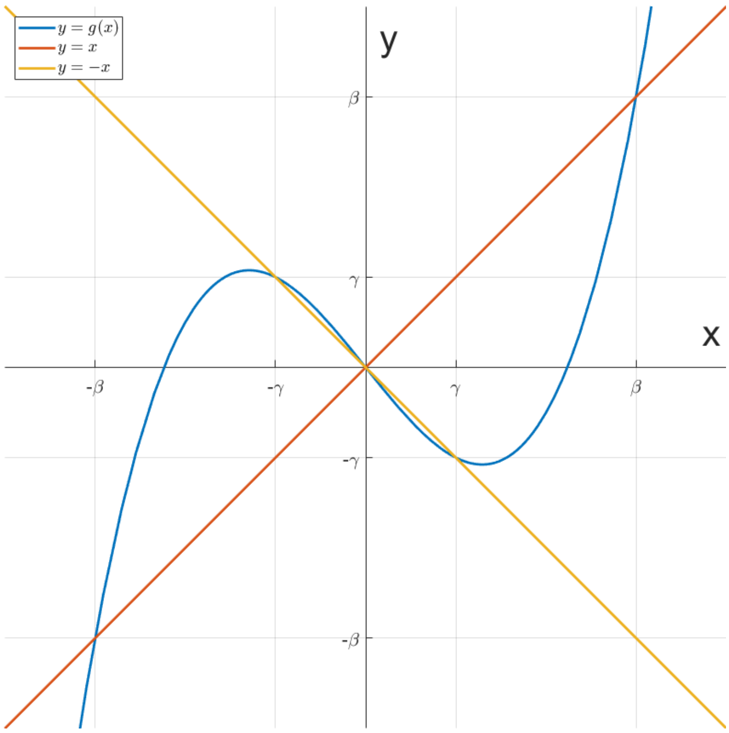


(a) Graph of f.



(b) Graph of g.

Figure 1

Short answers for exercise 1

1. a) For every x in $(-R, R)$

$$y''(x) = \sum_{n=2}^{+\infty} n(n-1)a_n x^{n-2}\,,$$

so that

$$\begin{aligned} xy''(x) &= \sum_{n=2}^{+\infty} n(n-1)a_n x^{n-1} \\ &= \sum_{n=1}^{+\infty} (n+1)na_{n+1}x^n\,, \end{aligned}$$

and so that

$$xy''(x) - y(x) = -a_0 + \sum_{n=1}^{+\infty} \big((n+1)na_{n+1} - a_n\big)x^n\,.$$

Thus, since $y(\cdot)$ is a solution of the differential equation, it follows that the previous power series is the zero function, which can happen only if all its coefficients are zero. It follows that $a_0 = 0$ and that, for every positive integer n,

$$(n+1)na_{n+1} - a_n = 0\,, \quad \text{or equivalently} \quad a_{n+1} = \frac{a_n}{(n+1)n}\,,$$

which is the intended recurrence relation.

b) By an immediate induction, it follows from the recurrence relation that, for every positive integer n,

$$a_n = \frac{a_1}{n!(n-1)!}\,.$$

c) The value of R is therefore $+\infty$ (ratio test).

2. The first terms of the power series of f read:

$$f(x) = 0 + x + \frac{x^2}{2} + \dots\,,$$

so that:

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1 \quad \text{and} \quad f''(0) = 1\,.$$

For the drawing, see Figure 2.

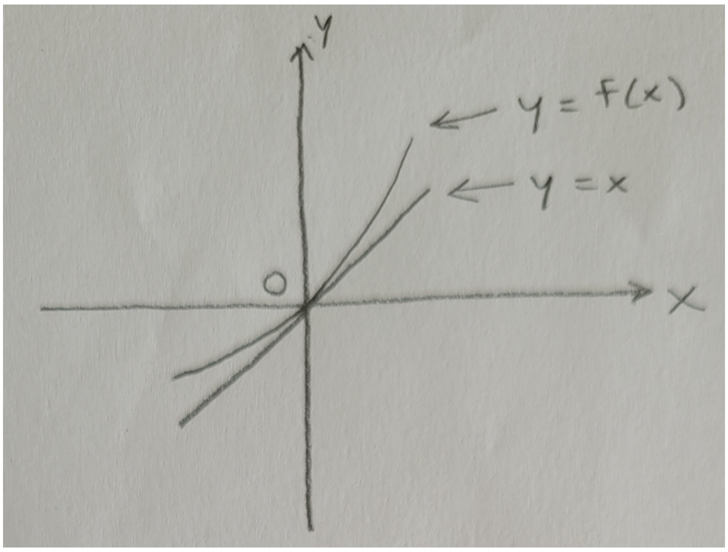


Figure 2: Graph of $x \mapsto f(x)$ close to $x = 0$.

a) The quantity $f(-1)$ is the sum of the series $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n}{n!(n-1)!} = \sum_{n=1}^{+\infty}(-1)^n b_n$, where $b_n = \dfrac{(-1)^n}{n!(n-1)!}$. Since $|b_n|$ is non-increasing with n and since b_n goes to 0 as n goes to $+\infty$, the alternating series test ensures that, for every positive integer N, if R_N denotes the remainder at rank N of this series, that is:

$$R_N = \sum_{n=N+1}^{+\infty} \frac{(-1)^n}{n!(n-1)!}\,,$$

then $|R_N| \leq b_{N+1}$. Besides,

$$1! \times 2! = 2 \quad \text{and} \quad 2! \times 3! = 2 \times 6 = 12 \quad \text{and} \quad 3! \times 4! = 6 \times 24 = 144\,,$$

so that $|R_3| \leq b_4 = \dfrac{1}{144} < \dfrac{1}{100}$. Thus,

$$\sum_{n=1}^{3} \frac{(-1)^n}{n!(n-1)!} = -1 + \frac{1}{2} - \frac{1}{12} = -\frac{7}{12}$$

approximates $f(-1)$ up to an error not exceeding 10^{-2}.

b) For every x in $[0, +\infty)$,

$$f'(x) = \sum_{n=1}^{+\infty} \frac{x^{n-1}}{\big((n-1)!\big)^2} = \sum_{n=0}^{+\infty} \frac{x^n}{(n!)^2} = 1 + x + \frac{x^2}{4} + \cdots \geq 1\,,$$

so that $f'(x)$ is positive (not smaller than 1), which proves the intended result.

c) Since $f(0) = 0$ and $f'(x)$ is not smaller than 1 for all x in $[0, +\infty)$, it follows from the Mean Value Theorem that, for all x in $[0, +\infty)$, $f(x)$ is not smaller than x. Other method: for all x in $[0, +\infty)$,

$$f(x) = x + \frac{x^2}{2} + \cdots \geq x\,.$$

It follows that $f(x)$ goes to $+\infty$ as x goes to $+\infty$.

Short answers for exercise 2

1. According to the inequalities given for the derivatives of f and g at their fixed points,

- 0 is an attractive fixed point, and $\pm\alpha$ are repulsive fixed points for f (position of the absolute value of the derivative of f at these fixed points, with respect to 1),
- 0 and $\pm\beta$ are repulsive fixed points for g.

2. Since $g(\gamma) = -\gamma$ and $g(-\gamma) = \gamma$, the sequence $(v_n)_{n\in\mathbb{N}}$ "alternates" between the values $\pm\gamma$: $v_{2n} = \gamma$ and $v_{2n+1} = -\gamma$; in other words, it is periodic of period two. Now, $|g'(\pm\gamma)| < 1$, this sequence can be qualified as "attractive" (it "attracts" nearby solutions, since applying the transformation g decreases the distances between two points and their images, close to this periodic sequence).

3. See Figure 3. The sequence $(u_n)_{n\in\mathbb{N}}$ goes to:

- $+\infty$ if $\alpha < u_0$,
- $-\infty$ if $u_0 < -\alpha$,
- 0 if $-\alpha < u_0 < \alpha$.

The sequence $(v_n)_{n\in\mathbb{N}}$ goes to:

- $+\infty$ if $\beta < v_0$,
- $-\infty$ if $v_0 < -\beta$.

If v_0 is in $(-\beta, 0) \cup (0, \beta)$, then two cases may occur:

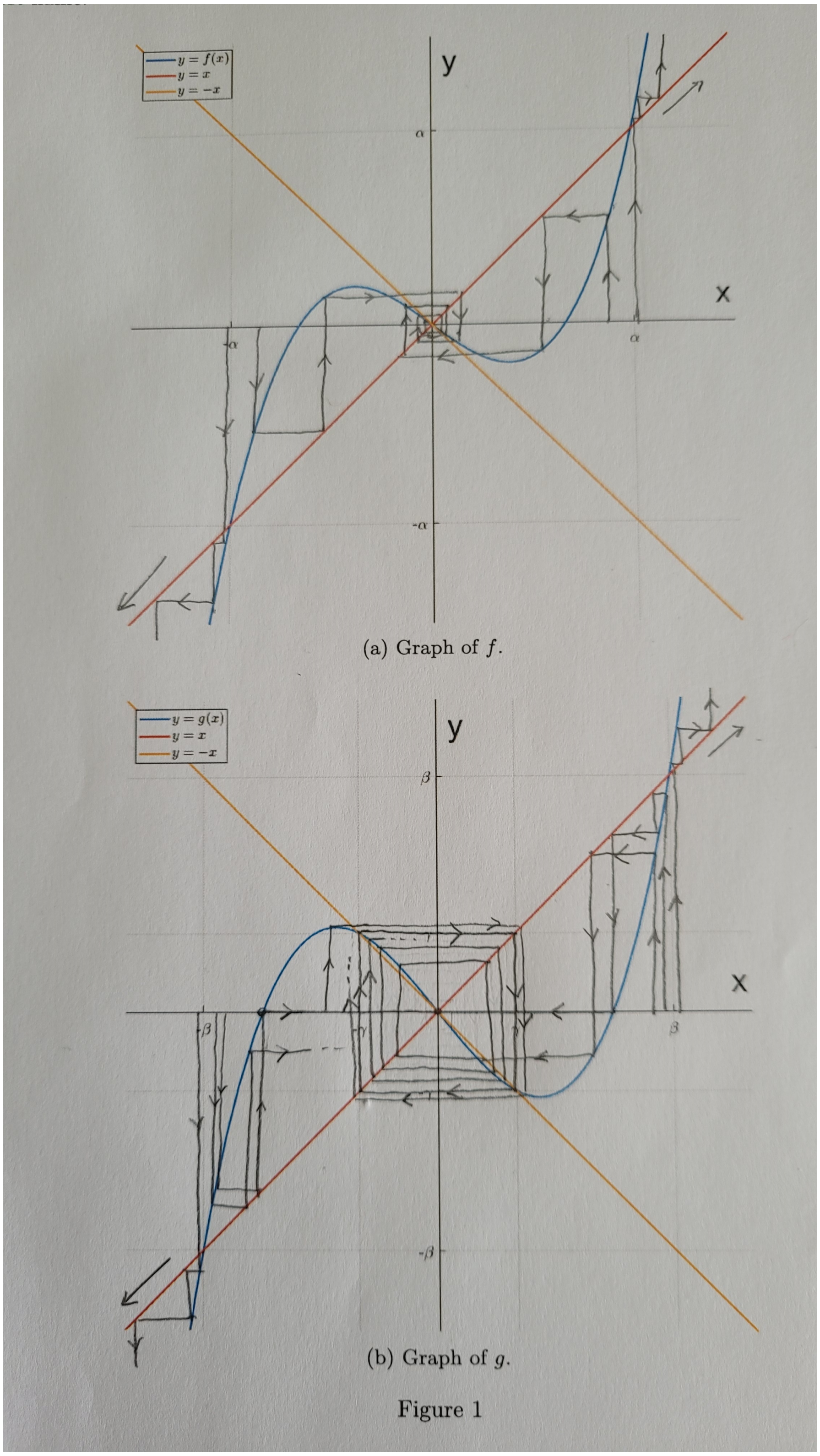


Figure 3: Behaviour of sequences (u_n) and (v_n), depending on the initial conditions u_0 and v_0.

1. either v_n equals 0 for some value of n, then the sequence is constant equal to 0 for all indices larger than n,
2. or the sequence $(v_n)_n$ gets closer and closer to a period-two sequence that oscillates between $-\gamma$ and γ. That is,
   - either v_{2n} goes to γ and v_{2n+1} goes to $-\gamma$,
   - or v_{2n} goes to $-\gamma$ and v_{2n+1} goes to γ.

4. If x is in $(\alpha, +\infty)$, then $f(x) > x > \alpha$ so that $f(x)$ is still in this interval. Thus, if u_0 is in this interval, u_n is in this interval for all n in $\mathbb{N}$, and the sequence u_n is increasing. Therefore, it either converges or goes to $+\infty$. If it were convergent, its finite limit ℓ would be a fixed point of f in $(\alpha, +\infty)$ (since f is continuous); since no such fixed point exists, the limit of $(u_n)_n$ is $+\infty$.

5. Since $w_{n+1} < w_n$, the sequence (w_n) is decreasing, and since w_n is nonnegative, this sequence converges towards some limit ℓ in $[0, \alpha)$. Since the function $x \mapsto |f(x)|$ is continuous, this limit must be a fixed point of $|f|$, thus a solution of $f(x) = \pm x$. According to the information given in the statement, it follows that ℓ must be equal to 0. Thus $(w_n)_n$ goes to 0, and as a consequence the same is true for $(u_n)_n$.

6. a) Since $f(0) = 0$, according to the Main Value Theorem, for every n in $\mathbb{N}$, there exists z_n between 0 and u_n such that

$$f(u_n) - f(0) = f'(z_n)(u_n - 0)\,, \text{ or in other words, } u_{n+1} = f'(z_n)u_n\,, \text{ thus } |u_{n+1}| = |f'(z_n)|\,|u_n|\,,$$

thus, by an immediate induction, for every positive integer n,

$$|u_n| = |u_0| \times \prod_{j=0}^{n-1} |f'(z_j)| = |u_0| \times |f'(z_0)| \times \cdots \times |f'(z_{n-1})|\,,$$

and since u_n goes to 0 as n goes to $+\infty$, the same is true for z_n.

b) It follows from the previous inequality that, for every positive integer n,

$$\frac{1}{n}\ln|u_n| = \frac{1}{n}\ln|u_0| + \frac{1}{n}\sum_{j=0}^{n-1}\ln|f'(z_j)|\,.$$

On the right-hand side of this equality, the first term goes to 0 as n goes to $+\infty$, and since z_j goes to 0 as j goes to $+\infty$, the terms in the sum get closer and closer to $\ln|f'(0)| = \ln(a) < 0$. It follows that the sum divided by n (the mean of the terms in the sum) converges to the same quantity $\ln|f'(0)|$ (this).

Short answers for exercise 3

1. For the definition of an inner product (symmetry, bilinearity, positive definiteness), see lecture notes. Here is a proof of the positive definiteness: for every P in $\mathbb{R}_3[X]$,

$$\varphi(P, P) = \int_{-1}^{1} P(t)^2\,dt\,;$$

Since P defines a continuous (actually, $\mathcal{C}^\infty$) function on $[-1, 1]$, if $\varphi(P, P)$ equals 0 then $P(t)$ must vanish on the whole interval $[-1, 1]$, thus P has infinitely many roots and is therefore (Fundamental Theorem of Linear Algerbra) the zero polynomial.

2. a) According to the Cauchy–Schwarz inequality (applied to the inner product φ and to some polynomial P and the constant polynomial 1),

$$\varphi(P,1)^2 \leq \varphi(1,1)\varphi(P,P)\,, \quad \text{or equivalently} \quad \left(\int_{-1}^{1} P(t)\,dt\right)^2 \leq 2\int_{-1}^{1} P(t)^2\,dt\,,$$

leading to the intended inequality.

b) According to lecture notes, this inequality is an equality if P and 1 are collinear vectors of $\mathbb{R}_3[X]$, that is if P is constant.

3. a) For every P in $\mathbb{R}_3[X]$, performing an integration by parts on the integral defining F,

$$\begin{aligned}\int_{-1}^{1}(t^3-t)P'(t)\,dt &= [(t^3-1)P(t)]_{-1}^{1} - \int_{-1}^{1}(3t^2-1)P(t)\,dt \\ &= -\int_{-1}^{1}(3t^2-1)P(t)\,dt \\ &= -\varphi(3X^2-1,P)\,,\end{aligned}$$

which proves the intended equivalence.

b) If $p_{F^\perp}$ denotes the orthogonal projection onto $F^\perp$ in $\mathbb{R}_3[X]$, then

$$\operatorname{dist}(Q,F) = \|p_{F^\perp}(Q)\| = \left|\frac{\varphi(Q,3X^2-1)}{\|3X^2-1\|}\right| = \frac{\left|\varphi(Q,3X^2-1)\right|}{\|3X^2-1\|} = \frac{\left|\varphi(Q,3X^2-1)\right|}{\sqrt{\varphi(3X^2-1,3X^2-1)}}\,.$$

By direct calculation,

$$\begin{aligned}\varphi(Q,3X^2-1) &= \int_{-1}^{1}(t+2t^2)(3t^2-1)\,dt \\ &= \int_{-1}^{1}(6t^4+3t^3-2t^2\ -t)\,dt \\ &= \left[\frac{6}{5}t^5+\frac{3}{4}t^4-\frac{2}{3}t^3-\frac{1}{2}t^2\right]_{-1}^{1} \\ &= \frac{12}{5}-\frac{4}{3} \\ &= \frac{16}{15}\,,\end{aligned}$$

and

$$\begin{aligned}\varphi(3X^2-1,3X^2-1) &= \int_{-1}^{1}(3t^2-1)(3t^2-1)\,dt \\ &= \int_{-1}^{1}(9t^4-6t^2+1)\,dt \\ &= \left[\frac{9}{5}t^5-\frac{6}{3}t^3+t\right]_{-1}^{1} \\ &= \frac{18}{5}-4+2 \\ &= \frac{8}{5}\,.\end{aligned}$$

Thus,

$$\operatorname{dist}(Q,F) = \frac{16\sqrt{5}}{15\sqrt{8}} = \frac{2\sqrt{8}}{3\sqrt{5}} = \frac{4\sqrt{2}}{3\sqrt{5}} = \frac{4\sqrt{10}}{15}\,.$$

Short answers for exercise 4

1. An orthogonal basis of F is given by vectors $v_1\ = u_1$ and $v_2\ = u_2\ +\lambda u_1$, where

$$\lambda = -\frac{\varphi(u_2,u_1)}{\varphi(u_1,u_1)} = -\frac{\int_0^\pi t\sin(t)\,dt}{\int_0^\pi \sin(t)\,dt} = -\frac{\pi}{2}\,,$$

so that $v_2 = u_2 - \frac{\pi}{2}u_1$, or equivalently $v_2(t) = t-\frac{\pi}{2}$.

2. The vector of F which is the closest to u_3 is the orthogonal projection, onto F, of u_3, that is:

$$p_F(u_3) = \frac{\varphi(u_3,v_1)}{\varphi(v_1,v_1)}v_1 + \frac{\varphi(u_3,v_2)}{\varphi(v_2,v_2)}v_2\,.$$

Now,

$$
\begin{aligned}
\varphi(u_3, v_1) &= \int_0^\pi \cos(t)\sin(t)\,dt = \frac{1}{2}\left[\sin(2t)\right]_0^\pi \\
&= 0\,, \\
\text{and} \quad \varphi(u_3, v_2) &= \int_0^\pi \cos(t)\left(t - \frac{\pi}{2}\right)\sin(t)\,dt \\
&= \int_0^\pi \frac{1}{2}\sin(2t)\left(t - \frac{\pi}{2}\right)\,dt \\
&= \left[-\frac{1}{2}\cos(2t)\left(t - \frac{\pi}{2}\right)\right]_0^\pi - \int_0^\pi \frac{1}{2}\sin(2t)\,dt \\
&= -\frac{\pi}{4} - \frac{\pi}{4} \\
&= -\frac{\pi}{2}\,, \\
\text{and} \quad \varphi(v_2, v_2) &= \int_0^\pi \left(t - \frac{\pi}{2}\right)^2 \sin(t)\,dt \\
&= \int_0^\pi \left(t^2 - \pi t + \frac{\pi^2}{4}\right)\sin(t)\,dt \\
&= \pi^2 - 4 - \pi^2 + \frac{\pi^2}{4} \times 2 \\
&= \frac{\pi^2}{2} - 4\,, \\
\text{so that} \quad p_F(u_3) &= \frac{\varphi(u_3, v_2)}{\varphi(v_2, v_2)} v_2 \\
&= \frac{-\pi/2}{\pi^2/2 - 4}\left(t - \frac{\pi}{2}\right) \\
&= -\frac{\pi}{\pi^2 - 8}\left(t - \frac{\pi}{2}\right)\,.
\end{aligned}
$$