INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — S3— Math Test #2 — 2h *December 8, 2016*

No documents, no calculators, no cell phones or electronic devices allowed but you may keep your pet blobfish for moral support.

All your answers must be fully justified, unless noted otherwise.

Exercise 1. Let $D = \mathbb{R}_+^* \times \mathbb{R}_+^*$ and define the mapping

$$\varphi : \begin{array}{ccc} D & \longrightarrow & D \\ (x, y) & \longmapsto & (x, y^2/x). \end{array}$$

Clearly, φ is of class C^∞; you don't have to justify this fact. We understand φ as a change of coordinates function, from the coordinates (x, y) to the coordinates (u, v) where $u = x$ and $v = y^2/x$.

1. a) Let $(x, y) \in D$. Compute the Jacobian matrix $J_{(x,y)}\varphi$ of φ at (x, y), and the determinant $\det J_{(x,y)}\varphi$.
   b) Show that φ is a bijection and determine φ^{-1}.
   c) Explain using only the fact that φ is a bijection and the result of Question 1a but *without using the explicit expression of* φ^{-1} why φ is a C^∞-diffeomorphism.
   d) Let $(u, v) \in D$. Compute the Jacobian matrix $J_{(u,v)}(\varphi^{-1})$ of φ^{-1} at (u, v), and check that $\left(J_{\varphi^{-1}(u,v)}\varphi\right)^{-1} = J_{(u,v)}(\varphi^{-1})$.
   e) Plot, *on the same figure,* "in the (x, y)-plane,"
      - several "u-coordinates", that is, curves of the form $u = C$ for several values of C,
      - several "v-coordinates", that is, curves of the form $v = C$ for several values of C.

      How can you see, on the figure, that φ is injective?
   f) Plot, *on the same figure,* "in the (u, v)-plane,"
      - several "x-coordinates", that is, curves of the form $x = C$ for several values of C,
      - several "y-coordinates", that is, curves of the form $y = C$ for several values of C.

      How can you see, on the figure, that φ is surjective?

2. Let $f : D \to \mathbb{R}$ be a function of class C^1 and define $g = f \circ \varphi^{-1}$ (so that $f = g \circ \varphi$). By composition, g is of class C^1 (and you don't have to justify this fact).
   a) Let $(x, y) \in D$. Compute the first order partial derivatives $\partial_1 f(x, y)$ and $\partial_2 f(x, y)$ of f at (x, y) in terms of the partial derivatives of g (at points that you will specify).
   b) Deduce, for $(x, y) \in D$, an expression of

      $$2x\partial_1 f(x, y) + y\partial_2 f(x, y)$$

      in terms of g and its first-order partial derivatives at points you will specify.
   c) We denote by (E) the following partial differential equation:

      (E) $$\forall (x, y) \in D,\ 2x\partial_1 f(x, y) + y\partial_2 f(x, y) = 4x f(x, y).$$

      and by (F) the following partial differential equation:

      (F) $$\partial_1 g = 2g.$$

      Show that

      $$f \text{ is a solution of (E)} \iff g \text{ is a solution of (F)}.$$
   d) Determine the general solution of class C^1 of (F) and deduce the general solution of class C^1 of (E).

Exercise 2. Let $n \in \mathbb{N}^*$. Define the function f as

$$f : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & \begin{cases} x^n \arctan\left(\dfrac{1}{y}\right) + y^n \arctan\left(\dfrac{1}{x}\right) & \text{if } xy \neq 0 \\ 0 & \text{if } xy = 0. \end{cases} \end{array}$$

1. Show that there exists $K \in \mathbb{R}_+^*$ such that[1]

$$\forall (x,y) \in \mathbb{R}^2,\ \left|f(x,y)\right| \leq K \left\|(x,y)\right\|_n^n.$$

2. Deduce that f is continuous at $(0,0)$.
3. Show that the partial derivatives $\partial_1 f(0,0)$ and $\partial_2 f(0,0)$ exist and determine their values.
4. Compute (if it exists! and the existence might depend on the value of n) the directional derivative $\nabla_{(1,1)} f(0,0)$ of f at $(0,0)$ in the direction $(1,1)$.
5. Deduce that if $n = 1$ then f is not differentiable at $(0,0)$.
6. Deduce, from Question 1 that if $n \geq 2$ then f is differentiable at $(0,0)$ and determine the differential $\mathrm{d}_{(0,0)} f$ of f at $(0,0)$.

Exercise 3. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a function of class C^1 and let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be a function of class C^2 such that

$$\forall (x,y) \in \mathbb{R}^2,\ J_{(x,y)}\varphi = \begin{pmatrix} \cos\bigl(g(x,y)\bigr) & -\sin\bigl(g(x,y)\bigr) \\ \sin\bigl(g(x,y)\bigr) & \cos\bigl(g(x,y)\bigr) \end{pmatrix}.$$

We denote by φ_1 and φ_2 the components of φ, that is, $\varphi_1 : \mathbb{R}^2 \to \mathbb{R}$ and $\varphi_2 : \mathbb{R}^2 \to \mathbb{R}$ are two functions of class C^2 and we have

$$\forall (x,y) \in \mathbb{R}^2,\ \varphi(x,y) = \bigl(\varphi_1(x,y), \varphi_2(x,y)\bigr).$$

1. From the Jacobian matrix of φ, read an expression of the first order partial derivatives of φ_1 and φ_2 in terms of g.
2. Show, using Schwarz' Theorem, that we must have:

$$\partial_1 g = \partial_2 g = 0$$

3. Deduce the form of φ.

Exercise 4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of class C^2 and let $u : \mathbb{R} \to \mathbb{R}$ be a function of class C^2. We define the function g as

$$g : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & f\bigl(x, u(xy)\bigr). \end{array}$$

1. Determine an explicit expression of $\partial_1 g$, $\partial_2 g$ in terms of f, u and their derivatives at well-chosen points that you will explicitly mention.
2. Determine an explicit expression of $\partial_{1,1}^2 g$, $\partial_{1,2}^2 g$, $\partial_{2,1}^2 g$ and $\partial_{2,2}^2 g$ in terms of f, u and their derivatives at well-chosen points that you will explicitly mention. In the process, check that $\partial_{1,2}^2 g = \partial_{2,1}^2 g$.
3. Check your answer to Question 1 in the special case

$$f : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & x^2 y, \end{array} \qquad u : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & x^2. \end{array}$$

[1] We recall that for $p \in [1, +\infty)$, the p-norm $\|\cdot\|_p$ defined on $\mathbb{R}^2$ by

$$\forall (u_1, u_2) \in \mathbb{R}^2,\ \left\|(u_1, u_2)\right\|_p = \left(|u_1|^p + |u_2|^p\right)^{1/p}$$

is a norm on $\mathbb{R}^2$.